CHAPTER 1

INTRODUCTION TO THE DYNAMICAL BEHAVIOUR OF ROTORS.

1.1: GENERAL CONSIDERATIONS.

The present chapter originates from the need to give the required basic concepts of rotordynamics to the reader. The intention is to prepare the reader to the following chapters by introducing concepts such as “whirl motion”, “self centring” and “critical speed”. In the references, the author gives a list of books and papers oriented towards the subjects of the dynamics of rotating machinery here discussed. In particular, the presentation of the subject in the present chapter is widely inspired by [1].
A rotor is a body suspended through a set of cylindrical bearings and rotating around an axis whose direction is fixed in the inertial space[1] ([1 - 3]). The part of the machine that does not rotate will be referred to stator. In the undeformed configuration, the rotation axis is well defined and fixed, and it coincides with one of the principal axis of inertia. Unfortunately, this is true only approximately, and the centre of mass of the suspended body does not coincide with the suspension point.

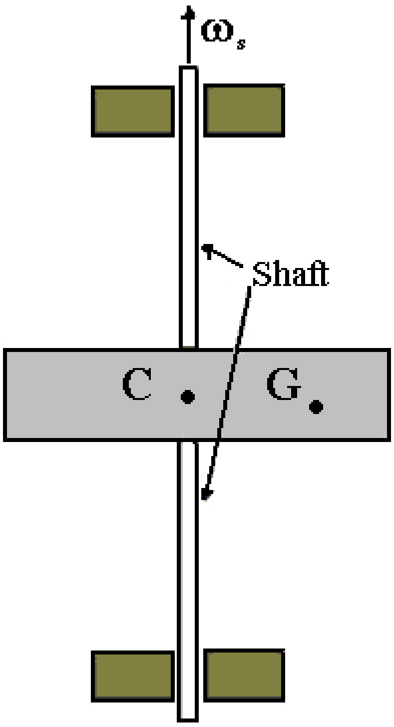

Figure 1.1: Longitudinal section of a rotor spinning at angular frequency ω_s. The centre of mass G does not coincide with the geometrical centre C of the disk. The shaft is elastic.

A simple model of rotor is shown in figure 1.1. The model is sketched in its undeformed configuration. $\vec{\omega}_s$ is the angular velocity about the axis of rotation and it is usually referred to as spin speed or spin frequency. G is the centre of mass of the suspended body. C is the suspension point of the body. In real world, the centre of mass G is not coincident with the

[1] ISO definition.

centre C; the distance between the two points $\vec{\varepsilon}$ is said eccentricity (or unbalance) and it can strongly affect the system. $\vec{\varepsilon}$ is a constant vector in the rotating frame fixed to the rigid body. In the non-rotating frame (inertial frame) the eccentricity rotates with angular velocity ω_s.
Forces due to the unbalance of the rotor itself can be described as vector rotating with the spin angular speed in the inertial reference frame. The spin velocities at which one of the forces acting on the rotor has a frequency coinciding with one of the natural frequencies[2] of the system are referred as critical speeds[3]. If the natural mode of the system is uncoupled to the forcing function no resonance occurs. However, the resonance takes place at critical speeds: the amplitude of the vibration grows linearly in time and the rotor can incur a failure. In particular, the coincidence of the spin speed with one of the natural frequencies of the rotor is very dangerous. The range of frequencies spanning from 0Hz to the first critical speed ω_{cr} is referred to as subcritical regime. Above ω_{cr}, the supercritical range starts. Working in the supercritical regime offers some advantages but at least one of the critical speeds must be crossed. Note that the concept of critical frequency can be defined only in the case of linear systems (or in the case of linearized systems)[4].
Rotors develop an unstable behaviour in certain velocity range. In the study of non-rotating damped linear system the amplitude of free oscillations decays in time because of dissipation due to damping. In the case of rotors, instead, the centrifugal field can cause a growth in time of the amplitude of free vibrations. The frequency range in which these self excited vibrations can develop is said instability range: the kinetic energy stored in the rotor is some orders of magnitude greater than the elastic potential energy the system can store before failure and it can sustain vibrations with increasing amplitude. The instability regions are always located in the supercritical range and working in the instability range is impossible. Critical speeds are very different from instability ranges: they occur at well-defined spin frequencies and can be passed if adequate damping is present.
If the time history of the system is expressed in the form $\underline{x}=\underline{x}_0 e^{st}$, the system is stable when the real part of the complex eigenvalues s is negative. We adopt the definition of stability introduced by Liapunov [4]: let us consider the vector $\underline{R}(t)$ in the state space, i.e. $\underline{R}(t)=(\underline{x}(t),\underline{v}(t))$, and use the expression $|\underline{R}(t)|$ for its norm. An equilibrium position $\underline{R}_0=(\underline{x}_0,\underline{0})$ is stable if $\forall\varepsilon>0\ \exists\delta>0$ such that the inequality $|\underline{R}(t)-\underline{R}_0|<\varepsilon$ holds $\forall t\in[0,\infty]$ if $|\underline{R}(t=0)-\underline{R}_0|<\delta$, i.e. if any trajectory starting within a circle of radius δ centred in the equilibrium point $\underline{R}_0=(\underline{x}_0,\underline{0})$ remains within a circle of radius ε for all values of time. The equilibrium position is asymptotically stable if $|\underline{R}(t)-\underline{R}_0|\to 0$ when $t\to\infty$.
Once the inertial reference frame is stated, the six equations of motion under the action of the generic force $\vec{F}$ and torque $\vec{M}$ can be written in the form:

$$\begin{cases} m\ddot{\vec{r}}=\vec{F} \\ \vec{M}=\dfrac{d\vec{L}}{dt} \end{cases} \tag{1.1}$$

[2] The natural frequencies are the solutions of the characteristic equation associated to the equation of motion of the system. The natural frequencies of a rotor can depend on the spin speed.
[3] The coincidence between the critical speed with the natural frequency of the undamped system is not a general feature of the rotors. It is a characteristic of those rotors in which the natural frequencies do not change with the spin velocity.
[4] Because only in this case the concept of natural frequency can be applied.

The assumptions of small unbalance and small displacement allow the equations of motion to be linearized. This chapter will be devoted to the study of the dynamic behaviour of simple rotors. In particular, some mathematical model that are not too complex will be discussed in order to introduce some of the most important characteristics of the rotors.

1.2: THE LINEAR JEFFCOTT ROTOR.

The simplest system that can be used to describe the dynamic behaviour of rotors is the so called Jeffcott rotor. The Jeffcott rotor consists of a point mass rigidly attached to a mass-less shaft. k is the stiffness of the elastic shaft and m the mass of the suspended body. A simple sketch of a Jeffcott rotor is depicted in figure 1.2: G is the point mass and C is the centre of the cross section of the shaft. The distance between the two points is the eccentricity $\vec{\varepsilon}=\overrightarrow{CG}$. The point G is always contained in the x-y plane[5]. $\vec{r}$ is the position vector of the point C with respect to the centre O of the undeformed shaft (O is also the origin of the (x,y,z) inertial reference frame). $\vec{r}_G = \vec{r} + \vec{\varepsilon}$ is the position vector of the point mass G. The line AOB coincides with the undeformed spin axis (shaft) along z direction; the line ACB is the deformed shaft. The assumptions of small unbalance and small displacement allow the equations of motion to be linearized. In the non-rotating frame (inertial frame) the eccentricity rotates with angular velocity ω_s ($\vec{\omega}_s = \omega_s \hat{z}$ is the angular spin speed of the rotor). $\varphi=\omega_s t$ is the angle between the eccentricity and the x axis of the inertial frame.

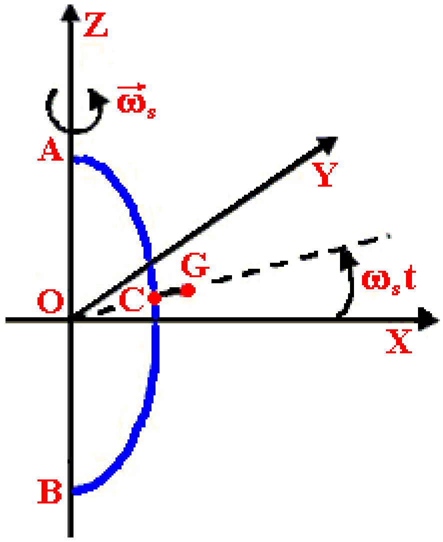

Figure 1.2: Sketch of the Jeffcott rotor. G is the point mass and C is the centre of the cross section of the shaft. The distance between the two points is the eccentricity $\vec{\varepsilon}=\overrightarrow{CG}$. $\vec{r}$ is the position vector of the point C with respect to the centre O of the undeformed shaft. $\vec{r}_G = \vec{r} + \vec{\varepsilon}$ is the position vector of the point mass G. The line AOB coincides with the undeformed spin axis (shaft); the line ACB is the deformed shaft.

Let us assume the displacements x and y of the point C as generalized coordinates (two degrees of freedom), i.e.:

$$\vec{r}=\overrightarrow{OC}=(x,y) \tag{1.2}$$

The position and velocity of point G are:

[5] This simplification is true in the limit of small displacements.

$$\vec{r}_G=\overrightarrow{OG}=\begin{pmatrix} x+\varepsilon\cos(\omega_s t) \\ y+\varepsilon\sin(\omega_s t) \end{pmatrix} \tag{1.3}$$

$$\dot{\vec{r}}_G=\begin{pmatrix} \dot{x}-\varepsilon\,\omega_s\sin(\omega_s t) \\ \dot{y}+\varepsilon\,\omega_s\cos(\omega_s t) \end{pmatrix} \tag{1.4}$$

The Lagrange function is then:

$$L=\frac{1}{2}m\left[\dot{x}^2+\dot{y}^2+\varepsilon^2\omega_s^2+2\varepsilon\omega_s\left(\dot{y}\cos(\omega_s t)-\dot{x}\sin(\omega_s t)\right)\right]-\frac{1}{2}k\left(x^2+y^2\right) \tag{1.5}$$

Making the assumption of constant spin speed (i.e. $\dot{\varphi}=\omega_s$), the following equations of motion are obtained through the lagrangean of the system:

$$\begin{cases} m\left[\ddot{x}-\varepsilon\,\omega_s^2\cos(\omega_s t)\right]+kx=0 \\ m\left[\ddot{y}-\varepsilon\,\omega_s^2\sin(\omega_s t)\right]+ky=0 \end{cases} \tag{1.6}$$

By introducing the complex coordinate $z' = x+jy$, the equation (1.7) is easily obtained from (1.6).

$$m\ddot{z}'+kz'=m\varepsilon\omega_s^2 e^{j\omega_s t} \tag{1.7}$$

The general solution of the homogeneous equation is then:

$$z'_h=z_1 e^{j\omega_{cr}t}+z_2 e^{-j\omega_{cr}t} \tag{1.8}$$

where $\omega_{cr}=\sqrt{k/m}$ is the critical speed of the system[6], often called whirl speed. Equation (1.8) shows that z' is a vector that rotates in the horizontal plane. The motion is the superimposition of a circular forward motion (also called forward whirl or direct whirl), occurring in the same direction as the spin angular velocity[7], and a backward motion (backward or indirect whirl). The resulting whirl motion can be circular, elliptic or rectilinear. The particular solution of equation (1.7) is:

$$z'_p=z_0 e^{j\omega_s t}=\varepsilon\frac{\omega_s^2}{\omega_{cr}^2-\omega_s^2}e^{j\omega_s t} \tag{1.9}$$

Equation (1.9) shows that the vector $\vec{r}$ (i.e. the point C) rotates in the plane with angular velocity ω_s, remaining in line with the vector $\overrightarrow{CG}=\varepsilon\left(\cos(\omega_s t),\sin(\omega_s t)\right)$, being z_0 the distance from the origin O of the reference frame[8]. Combining the equation (1.3) with (1.9), the position $\vec{r}_G$ of the point mass G reduces to:

$$\vec{r}_G=\varepsilon\frac{\omega_{cr}^2}{\omega_{cr}^2-\omega_s^2}\left(\cos(\omega_s t),\sin(\omega_s t)\right) \tag{1.10}$$

[6] It coincides with the natural frequency of the non rotating system.

[7] In this study the spin speed will be considered positive if anticlockwise.

[8] $z_0=\left|\vec{r}\right|$.

The value of the amplitude of r_G as a function of the spin speed is shown in figure 1.3. The point mass G rotates in the horizontal plane of the inertial frame with angular velocity ω_s. Instead, in a reference frame rotating with angular velocity $\vec{\omega}_s = \omega_s \hat{z}$, with the origin in O and z-axis coinciding with that of the preceding frame, the point mass G is in equilibrium. In subcritical regime, i.e. at spin speed lower than the critical one, the amplitude of $\vec{r}_G$ grows from ε to an infinite value (in coincidence with the critical speed ω_{cr}). In supercritical regime, (i.e. $\omega_s > \omega_{cr}$), G lies between O and C. The value of the amplitude is negative and it decreases with the spin angular velocity:

$$\vec{r} \cong -\vec{\varepsilon} \quad , \quad \vec{r}_G \cong -\left(\frac{\omega_{cr}}{\omega_s}\right)^2 \vec{\varepsilon} \cong 0 \quad \text{if } \omega_s >> \omega_{cr} \qquad (1.11)$$

Equation (1.11) means that in supercritical region there is a self-centring of the body on the rotation axis (i.e. G is practically coincident with O). The phenomenon is known as auto-centring in supercritical rotation[9]: the rotor rotates about its centre of mass instead of its geometrical centre.

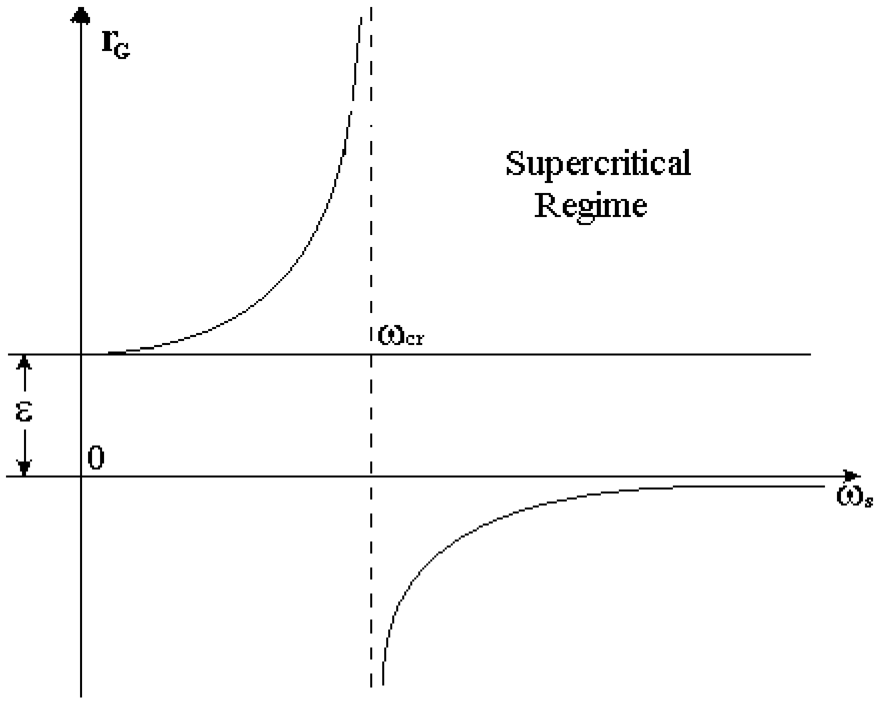

Figure 1.3 : Amplitude of the position vector $\vec{r}_G$ as a function of the spin speed. The amplitude grows from ε to an infinite value in coincidence with the critical speed ω_{cr}. In the supercritical range, i.e. at frequencies higher than the critical one, the value is negative and it decreases with the speed.

The final motion of the point C is the superimposition of a free whirl (circular, elliptic or linear) at frequency $\omega_{cr} = \sqrt{k/m}$ (see equation (1.8)) and a circular motion with angular velocity ω_s (see equation (1.9)).

[9]In order for the system to reach its equilibrium position on the opposite side with respect to the offset vector $\vec{\varepsilon}$, it must have two degrees of freedom. Indeed, it is well known that 1D systems are highly unstable if spinning at frequencies above the critical one. On this argument see section 1.11.

1.3: VISCOUS DAMPING.

Energy dissipation in rotating machines can cause the free motion to decay in time or increase ([1], [5 - 8]). When considering a damped system, it is important to distinguish between two different kinds of damping:

- the so called non-rotating damping, associated to the stationary parts of the apparatus
- the so called rotating damping, associated to the energy dissipation inside the rotor.

In the following we will show that non-rotating damping is stabilizing at any speeds. Instead, we will show that rotating damping reduces the amplitude of oscillations in case of subcritical rotation ($\omega_s<\omega_{cr}$), but it has destabilizing effects (whirling motions) when the rotor is in supercritical rotation. The force due to non-rotating damping is given in the non-rotating frame by (viscous damping model proportional to the velocity):

$$\vec{F}_{xy}^{NR}=-c_{NR}\left(\dot{x},\dot{y}\right) \tag{1.12}$$

where c_{NR} is the non-rotating damping coefficient. By using the complex notation, the force reduces to:

$$\vec{F}^{NR}=F_x^{NR}+jF_y^{NR}=-c_{NR}\dot{z}' \tag{1.13}$$

Let us now introduce the rotating reference frame (O,ξ,η,z) with the origin O and the z axis coinciding with that of the inertial frame. Axes ξ and η rotate in the x-y plane with angular velocity ω_s. The force due to rotating (viscous) damping is expressed in the rotating frame by:

$$\vec{F}_{\xi\eta}^{R}=-c_R\begin{pmatrix}\dot{\xi}\\ \dot{\eta}\end{pmatrix} \tag{1.14}$$

Introducing the complex position vector in the rotating frame $\zeta=\xi+j\eta$, it is readily obtained $\zeta=z'\,e^{-j\omega st}$. The derivative of the complex coordinate ζ is then:

$$\dot{\zeta}=\left(\dot{z}'-j\omega_s z'\right)e^{-j\omega_s t} \tag{1.15}$$

Combining equations (1.14) and (1.15), the force due to rotating damping in the rotating frame is obtained:

$$\vec{F}_{\xi\eta}^{R}=-c_R\dot{\zeta}=-c_R\left(\dot{z}'-j\omega_s z'\right)e^{-j\omega_s t} \tag{1.16}$$

In the inertial frame it reads:

$$\vec{F}_{xy}^{R}=-c_R\left(\dot{z}'-j\omega_s z'\right) \tag{1.17}$$

1.4: THE JEFFCOTT ROTOR WITH VISCOUS DAMPING.

By introducing the forces (1.13) and (1.17) at the right-hand side of the equation of motion (1.7), it follows[10]:

$$m\ddot{z}'+(c_R+c_{NR})\dot{z}'+(k-j\omega_s c_R)z'=m\varepsilon\omega_s^2 e^{j\omega_s t} \tag{1.18}$$

[10] If the angular frequency of the rotor is not constant, the equation that describes the motion of even a simple body is more complex and it is easier to perform the numerical integration. This problem will be shown in section 1.6.

The characteristic equation of the associated homogeneous equation gives the values of the λ eigenvalues:

$$\lambda = j\frac{c_R + c_{NR}}{2m} \pm \sqrt{\frac{4m\left(k - j\omega_s c_R\right) - \left(c_R + c_{NR}\right)^2}{4m^2}} \tag{1.19}$$

The eigenvalues (1.19) may be written in a more manageable manner by separating the real parts of the complex frequencies from the imaginary parts.
After introducing the three parameters $a = \left(4mk - (c_R + c_{NR})^2\right)/4m^2$, $b = -c_R\omega_s/m$ and $c = \left(c_R + c_{NR}\right)/2m$ the first eigenvalue is obtained as[11]:

$$\lambda_1 = -\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + j\left(c + \sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right) \tag{1.20}$$

It has a negative real part, namely $\Re e(\lambda_1) < 0$[12]. Hence, the corresponding motion $z' = z_1 e^{-\Im m(\lambda_1)t} e^{j\Re e(\lambda_1)t}$ is a backward whirl mode. The imaginary part is always positive in the whole range of frequency. It corresponds then to a damped backward whirl with amplitude decreasing in time with exponential law. The centre of mass of the rotor spiralises toward the centre of rotation, i.e. the origin O of the reference frame. This whirl damps out quickly and has little practical interest. The second eigenvalue is:

$$\lambda_2 = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + j\left(c - \sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right) \tag{1.21}$$

It has a positive real part and corresponds to a forward whirl motion. The imaginary part can be either positive or negative and the corresponding whirl can be either damped or excited. The condition for stability in terms of the sign of λ_2 is $\Im m(\lambda_2) > 0$. With simple algebra it can be shown to be:

$$\omega_s < \omega_{cr}\left(1 + c_{NR}/c_R\right) \tag{1.22}$$

If only rotating damping is present the motion is unstable in whole supercritical regime. In case of highly supercritical rotation the condition for stability means:

$$c_{NR} > c_R\, \omega_s/\omega_{cr} \tag{1.23}$$

Equation (1.23) means that non-rotating damping has a stabilizing effect on the rotor. The condition is easy to fulfil due to the very low level of the rotating damping of real rotors. The particular integral of the non-homogeneous equation (1.18) is connected to the presence of the unbalance and can be written in the form:

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[11] $\sqrt{a + jb} = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + j\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$

[12] In the limit $\omega_s \gg \sqrt{(k/m)}$ and $Q \gg 1$ it is easily found: $\Re e(\lambda_1) = -\omega_{cr}$ and $\Re e(\lambda_2) = \omega_{cr}$.

$$z'_p = z_0 e^{j\omega_s t} = \varepsilon \frac{\omega_s^2}{\omega_{cr}^2 - \omega_s^2 + j\omega_s c_{NR}/m} e^{j\omega_s t} \qquad (1.24)$$

Starting from equation (1.24) the motion of the centre of mass[13] is readily obtained. In particular, we can write the value of the amplitude as a function of the spin speed:

$$\left| z_G(j\omega_s) \right| = \varepsilon \left| \frac{\omega_s^2}{\omega_{cr}^2 - \omega_s^2 + j\dfrac{\omega_s c_R}{m}} + 1 \right| \qquad (1.25)$$

By introducing the dimensionless parameter

$$\gamma_R = c_R/(2m\omega_{cr}) = 1/Q \qquad (1.26)$$

equation (1.25) can be written as:

$$\left| z_G(j\omega_s) \right| = \varepsilon \sqrt{1 + 4\gamma_R^2 \frac{\omega_s^2}{\omega_{cr}^2}} \Bigg/ \sqrt{\left(1 - \frac{\omega_s^2}{\omega_{cr}^2}\right)^2 + \left(2\gamma_R \frac{\omega_s}{\omega_{cr}}\right)^2} \qquad (1.27)$$

Figure 1.4 shows the dependence of the amplitude of z_G (1.27) as a function of the angular spin frequency.

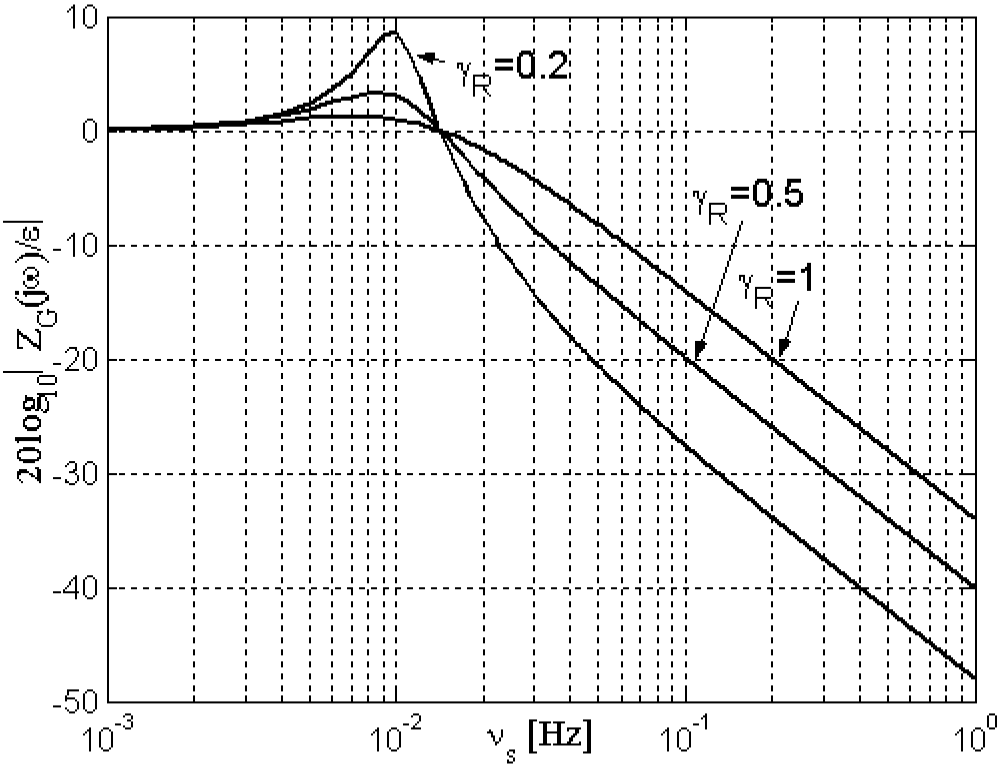

Figure 1.4: Dimensionless amplitude of the distance of the point mass G from the rotation axis for three different values of rotating damping.

13 $z_G = x_G + jy_G = z' + \varepsilon e^{j\omega_s t}$

In supercritical regime it decreases with the spin angular velocity and vanishes in the limit $\omega_s \rightarrow \infty$, i.e. there is a self-centring of the suspended body on the rotation axis. If the stability condition (1.23) is satisfied, the general solution of the equation (1.18) can be considered as the superimposition of a backward whirl motion (the solution corresponding to the eigenvalue (1.20)), which decays in time, a forward motion, with decreasing amplitude (the solution corresponding to (1.21)) and a circular motion at the spin frequency with constant amplitude (corresponding to (1.24)). There is a self-centring of the point mass G, instead, on the rotation axis, at distance $\sim \varepsilon\, \omega_{cr}^2 / \omega_s^2 \ll \varepsilon$.

1.5: STRUCTURAL DAMPING.

Many materials, when subjected to cyclic loading, show a behaviour that can be described in terms of a hysteresis cycle. In these cases the damping is called "structural". Structural damping is due to the relative motions of different parts in the material when subject to deformations. Let us introduce the complex stiffness $k^*=k^R+jk^I$ ([1], [9]): the real part k^R is linked with the elastic stiffness of the material, while the imaginary part k^I is connected to the damping. $\gamma_k=k^I/k^R$ is the loss factor. The expression of the complex stiffness may be written as $k^*=k(1+j\gamma_k)$. For simplicity, we consider a model with only one degree of freedom, that is simple but demonstrates, at least qualitatively, the behaviour of more complex systems: on the point mass m can act a force function of time f(t) and the supporting point can move in the x direction. The equation of motion can be solved in the frequency domain:

$$k_{din}\, X(j\omega)=F(j\omega) \tag{1.28}$$

In equation (1.28) we have introduced the dynamic stiffness of the system with structural damping:

$$k_{din} = -m\omega^2 + k(1+j\gamma_k) \tag{1.29}$$

The dynamic stiffness is a function of the forcing frequency, and in case of damped system, it is complex. The complex frequency of the free oscillations can be obtained by equating to zero the dynamic stiffness:

$$\omega_c = \sqrt{k/m}\sqrt{1+j\gamma_k} = \omega_n\sqrt{1+j\gamma_k} \tag{1.30}$$

where ω_n is the natural frequency of the undamped system.
After some simple algebra[14], the real and the imaginary parts of the complex frequency can be easily separated:

$$\omega_c = \omega_n\sqrt{\frac{1+\sqrt{1+\gamma_k^2}}{2}} + j\omega_n\sqrt{\frac{-1+\sqrt{1+\gamma_k^2}}{2}} \tag{1.31}$$

The loss factor is typically very small (γ_k<<1); in this limit it follows:

$$\Re e(\omega_c) \approx \omega_n \tag{1.32}$$

[14] $\sqrt{2(a+jb)} = \sqrt{a+\sqrt{a^2+b^2}} + j\sqrt{-a+\sqrt{a^2+b^2}}$

$$\Im m(\omega_c) \approx \omega_n \frac{\gamma_k}{2} \tag{1.33}$$

Relations (1.32) and (1.33) show that the frequency shift due to the presence of structural damping is negligible for lightly damped systems.
The ratio between the elastic stiffness and the dynamic stiffness is usually referred to as the frequency response H of the system. The expressions for the real and imaginary parts, its amplitude and phase are:

$$\Re e(H(j\omega)) = k \frac{k - m\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + k^2\gamma_k^2}} \tag{1.34}$$

$$\Im m(H(j\omega)) = -\frac{k^2\gamma_k}{\sqrt{\left(k - m\omega^2\right)^2 + k^2\gamma_k^2}} \tag{1.35}$$

$$\left|H(j\omega)\right| = \frac{k}{\sqrt{\left(k - m\omega^2\right)^2 + k^2\gamma_k^2}} = \frac{1}{\sqrt{\left(1 - \omega^2/\omega_n^2\right)^2 + \gamma_k^2}} \tag{1.36}$$

$$\Phi(j\omega) = \mathrm{arctg}\left(-\frac{k\gamma_k}{k - m\omega^2}\right) = \mathrm{arctg}\left(-\gamma_k \Big/ \left(1 - \omega^2/\omega_n^2\right)\right) \tag{1.37}$$

Figure 1.5 shows the amplitude and phase of H as function of the forcing frequency for different values of the loss factor.

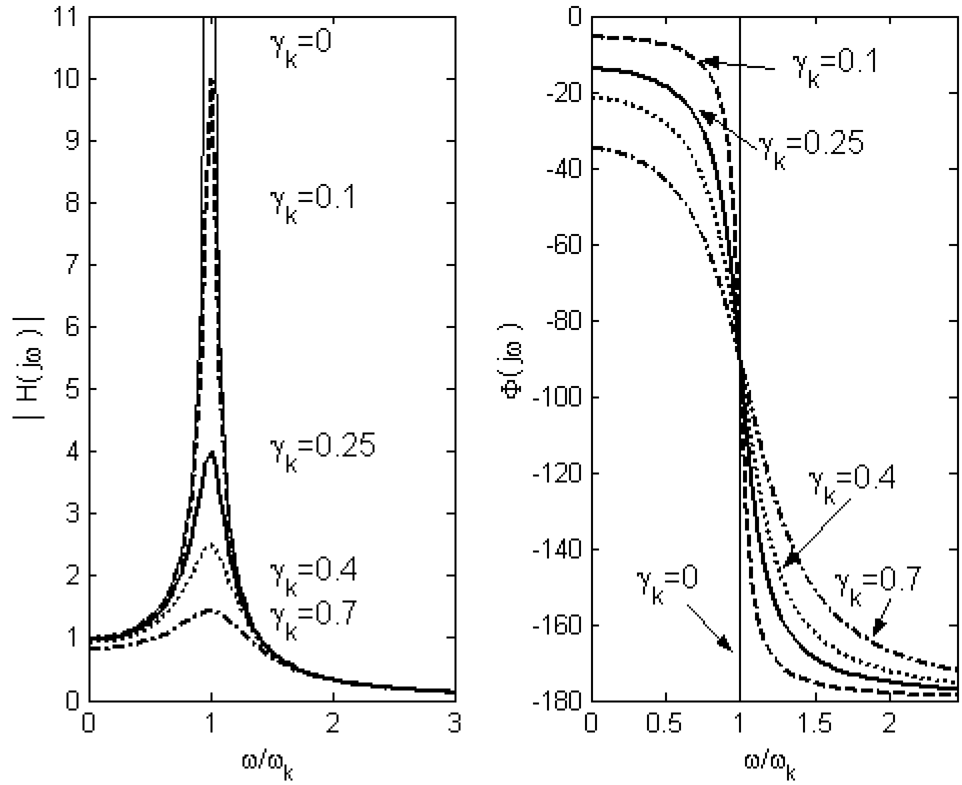

Figure 1.5: Amplitude and phase of H(jω) as function of the forcing frequency. For different values of the loss factor.

The quality factor of the system with structural damping is defined as:

$$Q = \left|H(j\omega)\right|_{max} = \frac{1}{\gamma_k} \tag{1.38}$$

Structural damping is then a form of linear damping and it can be assimilated with viscous damping through the equivalent viscous damping coefficient:

$$c_{eq} = k\gamma_k / \omega \tag{1.39}$$

In equation (1.39) ω is the frequency at which the material goes through the hysteresis cycle. By inserting the relation (1.38) into the (1.39), it follows:

$$c_{eq} = \frac{k}{\omega Q} \tag{1.40}$$

On the basis of experiences with many rotating machines it is concluded that friction inside rotating parts (the suspensions) is essentially of structural nature. Hence, a structural damping model is usually better suited for the rotating damping. Note that the frequency at which the hysteresis cycle is gone through is now $|\omega_n-\omega_s|$ with ω_s the spin speed:

$$c_R = \frac{k}{Q\left|\omega_n - \omega\right|} \text{ [15]} \tag{1.41}$$

In case of highly supercritical rotation, the relation (1.41) can be simplified in $c_R \approx k/Q\omega$, i.e. energy is dissipated at the high spin angular velocity and not at the low natural frequency.

1.6: ACCELERATING JEFFCOTT ROTOR.

If the spin speed is constant, the terms in $\dot{\omega}_s$ in equation (1.18) are neglected. However, when the rotational speed is not constant, the angle between the two reference frames (inertial and rotating; see section 1.2) can not be written as $\varphi=\omega_s t$. φ is now a third generalized coordinate linked with the rotation about the z axis. A driving torque M_z is assumed to be applied to the shaft of the rotor (torsionally stiff) and the lagrangean of the system is expressed by the relation:

$$\mathcal{L} = \frac{1}{2}m\left[\dot{x}^2+\dot{y}^2+\varepsilon^2\dot{\varphi}^2+2\varepsilon\dot{\varphi}\left(\dot{y}\cos(\varphi)-\dot{x}\sin(\varphi)\right)\right]-\frac{1}{2}k\left(x^2+y^2\right)+\frac{1}{2}J_z\dot{\varphi}^2 \tag{1.42}$$

The equations of motion are determined from (1.42) in the standard manner:

$$m\ddot{z}'+(c_R+c_{NR})\dot{z}'+(k-j\dot{\varphi}c_R)z'=m\varepsilon\left(\dot{\varphi}^2 - j\ddot{\varphi}\right)e^{j\varphi} \text{ [16]} \tag{1.43}$$

$$\left(J_z + m\varepsilon^2\right)\ddot{\varphi} + m\varepsilon\left(-\ddot{x}\sin(\varphi) + \ddot{y}\cos(\varphi)\right) = M_z \text{ [17]} \tag{1.44}$$

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[15] For the non-rotating damping the equivalent coefficient is $c_{NR} = k/Q\omega_n$.

[16] If the spin speed is constant, equation (1.18) is easily obtained from (1.43) by performing the substitutions $\varphi\rightarrow\omega_s t$, $d\varphi/dt \rightarrow\omega_s$, $d^2\varphi/dt^2 \rightarrow 0$.

In this case it is better to introduce a rotating frame (which, however, does not rotate at constant speed). The position, velocity and acceleration of point C can be expressed as functions of complex coordinate ζ:

$$z' = \zeta e^{j\varphi} \tag{1.45}$$

$$\dot{z}' = \left(\dot{\zeta} + j\dot{\varphi}\zeta\right)e^{j\varphi} \tag{1.46}$$

$$\ddot{z}' = \left(\ddot{\zeta} + j\ddot{\varphi}\zeta + 2j\dot{\varphi}\dot{\zeta} - \varphi^2\zeta\right)e^{j\varphi} \tag{1.47}$$

By introducing the relations (1.45), (1.46) and (1.47) into the equations (1.43) and (1.44) it follows:

$$m\ddot{\zeta}+(c_R+c_{NR}+2mj\dot{\varphi})\dot{\zeta}+(k+j\dot{\varphi}c_{NR}+j\ddot{\varphi}-\varphi^2)\zeta=m\varepsilon\left(\dot{\varphi}^2-j\ddot{\varphi}\right) \tag{1.48}$$

$$\left(J_z+m\varepsilon^2\right)\ddot{\varphi}+m\varepsilon\left(\ddot{\eta}+2\dot{\varphi}\dot{\xi}+\ddot{\varphi}\xi-\dot{\varphi}^2\eta\right)=M_z\ ^{18} \tag{1.49}$$

When the time history of the driving torque is known, equations (1.48) and (1.49) can be solved by performing a numerical integration.
We want, now, to evaluate the torque needed to operate at constant speed ω_s. In this limit ($\ddot{\varphi}=0$), equations (1.48) coincide with equations (1.18) and the displacement ζ is constant at the value expressed by (1.24), namely:

$$\zeta_{\ddot{\varphi}=0}=\varepsilon\omega_s^2\Big/\left(\omega_{cr}^2-\omega_s^2+j\omega_s c_{NR}/m\right) \tag{1.50}$$

By stating $\ddot{\varphi}=0, \dot{\varphi}=\omega_S, \varphi=\omega_S t$, equation (1.49) can be written in the form:

$$-m\varepsilon\omega_s^2\eta_{\ddot{\varphi}=0}=M_z\ ^{19} \tag{1.51}$$

By combining equations (1.50) with (1.51), the torque needed to operate at constant speed ω_s is obtained:

$$M_z=\frac{c_{NR}\varepsilon^2\omega_s^5}{\left(\omega_{cr}^2-\omega_s^2\right)^2+\dfrac{\omega_s^2 c_{NR}^2}{m^2}}\ ^{20} \tag{1.52}$$

---

[17] The equation (1.44) is obtained in the standard manner: $\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\varphi}}-\dfrac{\partial L}{\partial\varphi}=M_Z$.

[18] In sections 1.2 and 1.3 we have introduced the complex variables $z'=x+jy$ and $\zeta=\xi+j\eta$. After some simple algebra, it follows: $\zeta=z'e^{-j\varphi}$ and $\Im m(\ddot{z}'e^{-j\varphi})=-\ddot{x}\sin(\varphi)+\ddot{y}\cos(\varphi)$. Equation (1.44) can be written as $(J_z+m\varepsilon^2)\ddot{\varphi}+m\varepsilon\Im m\left(\ddot{z}'e^{-j\varphi}\right)=M_z$. By noticing that $\Im m\left(\ddot{z}'e^{-j\varphi}\right)=\ddot{\eta}+2\dot{\varphi}\dot{\xi}+\ddot{\varphi}\xi-\dot{\varphi}^2\eta$, equation (1.49) is readily obtained.

[19] We have equated to zero all derivatives of the generalized coordinates: $\ddot{\varphi}=\dot{\xi}=\ddot{\xi}=\dot{\eta}=\ddot{\eta}=0$.

[20] $\eta_{\ddot{\varphi}=0}=\Im m\left(\zeta_{\ddot{\varphi}=0}\right)=-\dfrac{c_{NR}\varepsilon\omega_s^3/m}{\left[\left(\omega_{cr}^2-\omega_s^2\right)^2+\dfrac{\omega_s^2c_{NR}^2}{m^2}\right]}$

Let us introduce the loss factors:

$$\gamma_R = \frac{c_R}{2\sqrt{km}}; \; \gamma_{NR} = \frac{c_{NR}}{2\sqrt{km}} \qquad (1.53)$$

The driving torque may be written in terms of non-dimensional parameters:

$$M_z = 2\gamma_{NR} k\varepsilon^2 \frac{\omega_s^5 / \omega_{cr}^5}{\left(1-\omega_s^2/\omega_{cr}^2\right)^2 + 4\gamma_{NR}^2\, \omega_s^2/\omega_{cr}^2} = 2k\varepsilon^2 M_z^* \qquad (1.54)$$

Figure 1.6 shows the non-dimensional driving torque M_Z^* as a function of the ratio ω_s/ω_{cr}.

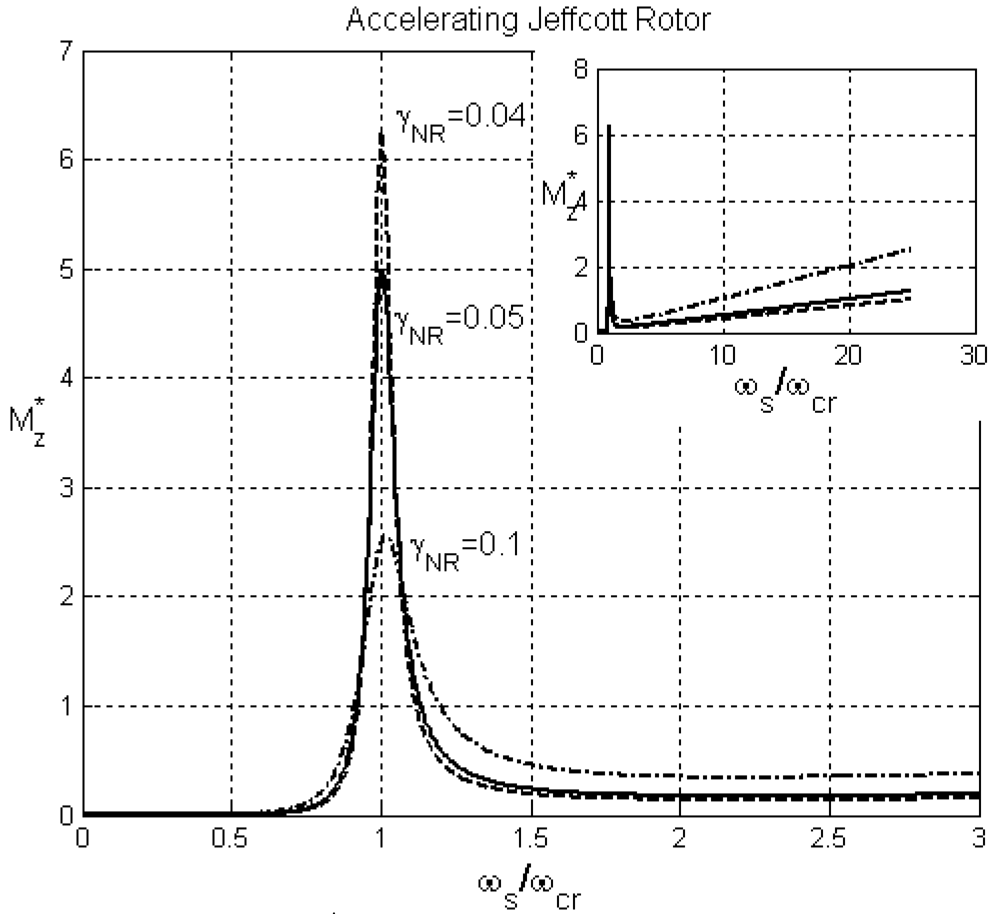

Figure 1.6: Non-dimensional torque M_Z^* as a function of the ratio ω_s/ω_{cr}. The torque has a peak at the critical speed. Inset: value of the torque at high spin speed.

The torque has a peak at the critical frequency:

$$M_{cr} = M_z(\omega_s = \omega_{cr}) = \frac{k\varepsilon^2}{2\gamma_{NR}} \qquad (1.55)$$

and, in case of highly supercritical rotation, the torque needed to operate at constant speed grows linearly with the spin angular velocity:

$$\lim_{\omega_s \to \infty} M_z = 2k\varepsilon^2 \gamma_{NR}\, \omega_s \qquad (1.56)$$

Note that M_{cr} is also the smaller torque needed to operate at speed higher than the critical speed.

1.7: COUPLED ROTORS.

In order to investigate the properties of coupled rotors ([10 - 11]), we examine the simple problem of two bodies connected by dissipative springs. Two concentric, co-axial, hollow cylinders with mass $m_1=m_2=m$, weakly coupled by dissipative mechanical suspensions with elastic stiffness k, rotates around their symmetry axis (z axis) at constant spin speed ω_s higher than the natural frequency of the system (supercritical regime). The rotation is counter-clockwise[21].

In supercritical rotation mechanical suspensions are known to undergo deformation (and therefore to dissipate energy) at the spin frequency. Energy dissipation makes the spin rate to decrease, together with the spin angular momentum. Since the total angular momentum must be conserved, the bodies develop a whirl motion of increasing amplitude around each other at a frequency close to the natural differential[22] one due to the coupling. In figure 1.7 a sketch of the two coupled cylinders is shown. Since the springs are very weak and their masses are negligible compared to the mass of the rotor, they will be obliged to follow the motion of the attachment points which rotate at ω_s around the centre of mass of the respective test mass. The centres of mass of the springs will rotate around O at ω_s. When the springs are going from position 1 to position 3 in figure 1.7, they will be forced to expand by $4r_w$ (r_w is the radius of the whirl motion), and when going from position 3 to 1 to contract by the same amount.

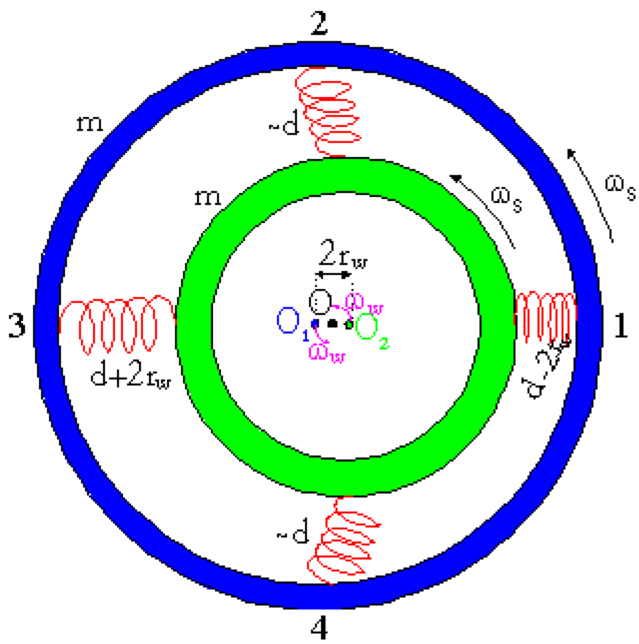

Figure 1.7: Simple model of a system made of two bodies of mass m, coupled by weak springs. Both bodies are spinning at the same angular velocity ω_s around their respective centres of mass O_1 and O_2. O_1 and O_2 are whirling around the centre of mass O of the whole system at the natural frequency $\omega_w=\omega_n$.

---

[21] This is a simplified model for the system spacecraft – PGB which will be studied in chapters 7,8 and 9.

[22] Natural differential frequency: the cylinders' centres of mass move within the horizontal plane in opposition of phase while their symmetry axis remains aligned with the vertical z

The centres of mass O_1 and O_2 of the two cylinders rotate around O with angular frequency $\omega_w=\omega_n$ (ω_n is the natural frequency of the system). After the spring, starting from position 1, has completed one turn in the time $T_s=2\pi/\omega_s$, the whirling motion will have displaced position 1 by an angle $\pm 2\pi\omega_w/\omega_s$ (the sign + refers to the forward whirling and the sign – to the backward one). Therefore, in order to reach again the position 1 of maximum contraction, the spring takes a time slightly different from T_s. This means that each spring is forced to oscillate at the frequency $\omega_s \pm \omega_w$. As a consequence, by considering the dissipation of the whole system as expressed by the quality factor *Q*, the coefficient of rotating damping is given as $c_R=k/(Q|\omega_s \pm \omega_w|)$ in agreement with equation (1.41). Let us consider $\vec{r}_1$ the position vector of the cylinder 1, $\vec{r}_2$ the position vector of body 2, $\vec{\varepsilon}$ (eccentricity) the vector locating the suspension point of the spring with respect to the centre of the outer body 2 (see figure 1.8).

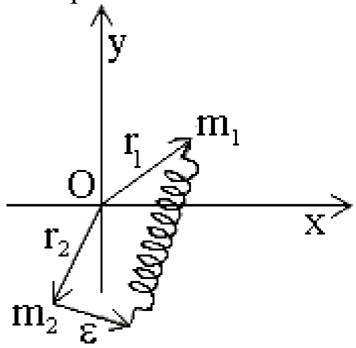

Figure 1.8: Simplified model of the system. $\vec{r}_1$ is the position vector of the cylinder 1, $\vec{r}_2$ the position vector of body 2, $\vec{\varepsilon}$ (eccentricity) the vector locating the suspension point of the spring with respect to the centre of the outer body 2.

In the inertial reference frame the equations of motion of the centres of mass are:

$$\begin{cases} m_1\ddot{\vec{r}}_1=-k(\vec{r}_1-\vec{r}_2+\vec{\varepsilon})-c_R\left(\dot{\vec{r}}_1-\dot{\vec{r}}_2-\vec{\omega}_s\times(\vec{r}_1-\vec{r}_2)\right) \\ m_2\ddot{\vec{r}}_2=-k(\vec{r}_2-\vec{r}_1-\vec{\varepsilon})-c_R\left(\dot{\vec{r}}_2-\dot{\vec{r}}_1-\vec{\omega}_s\times(\vec{r}_2-\vec{r}_1)\right) \end{cases} \tag{1.57}$$

where the rotating damping (1.17) has been introduced. By defining the relative position vector $\vec{\xi}=\vec{r}_2-\vec{r}_1=(\xi_1,\xi_2)$, the reduced mass $m_r=(m_1m_2)/(m_1+m_2)$ and the natural frequency $\omega_n^2=k/m_r$, we can write the equation for the relative motion of the cylinders:

$$\ddot{\vec{\xi}}=-\omega_n^2\left(\vec{\xi}-\vec{\varepsilon}\right)-\frac{c_R}{m_r}\left(\dot{\vec{\xi}}-\vec{\omega}_s\times\vec{\xi}\right) \tag{1.58}$$

By noticing that for a highly spinning rotor the coefficient of rotating damping takes the value $c_R = m_r\,\omega_n^2/(Q\omega_s)$ and being the vector $\vec{\varepsilon}=\vec{\varepsilon}(t)=\varepsilon(\cos(\omega_s t),\sin(\omega_s t))$ a rotating vector in the inertial reference frame (it is fixed in the reference frame co-rotating with the rotor), it follows:

$$\ddot{\vec{\xi}}+\frac{\omega_n^2}{Q\omega_s}\dot{\vec{\xi}}+\left[\omega_n^2\vec{\xi}-\frac{\omega_n^2}{Q\omega_s}\left(\vec{\omega}_s\times\vec{\xi}\right)\right]=\omega_n^2\vec{\varepsilon}(t) \tag{1.59}$$

Equation (1.59) can be written in terms of the complex coordinate $z'=\xi_1+j\xi_2$:

$$\ddot{z}'+\frac{\omega_n^2}{Q\omega_s}\dot{z}'+\left(\omega_n^2-j\frac{\omega_n^2}{Q}\right)z'=\omega_n^2\varepsilon e^{j\omega_s t} \tag{1.60}$$

Before finding the solutions of equation (1.60), let us replace the right-hand side of this equation with the forcing function[23] $f(t)=\alpha e^{j\omega t}$. The transfer function is now:

$$H(j\omega)=\frac{Z(j\omega)}{F(j\omega)}=1\Big/\left[\left(\omega_n^2-\omega^2\right)+j\left(\frac{\omega-\omega_s}{\omega_s}\right)\frac{\omega_n^2}{Q}\right] \tag{1.61}$$

In the particular case of equation (1.60) (i.e. when $\omega=\omega_s$, $\alpha=\omega_n^2\varepsilon$ and the forcing function $f(t)=\omega_n^2\varepsilon e^{j\omega_s t}$) the transfer function (1.61) can be written as:

$$H(j\omega_s)=1\Big/\left(\omega_n^2-\omega_s^2\right) \tag{1.62}$$

The particular integral of the non-homogeneous equation (1.59) can be readily obtained from (1.62):

$$\vec{\xi}_\varepsilon(t)=\frac{\omega_n^2}{\left(\omega_n^2-\omega_s^2\right)}\vec{\varepsilon}(t) \tag{1.63}$$

Equation (1.10) obtained in the case of the Jeffcott rotor and equation (1.63) are very similar. In supercritical regime, the amplitude of the vector $\vec{\xi}_\varepsilon(t)$ decreases with the spin speed. The equation (1.60) is like equation (1.18) and its solution are obtained in the same manner. In the limit of highly supercritical rotation $\omega_s >> \omega_n$ and high quality factor $Q>>1$, its eigenvalues are:

$$\lambda_{1,2}=\mp\omega_n\left[1-j/\left(2Q\right)\right] \quad ^{24} \tag{1.64}$$

Having the eigenvalues (1.64), the general solution of the homogeneous equation associated with (1.59) may be easily written in the form:

$$\vec{\xi}_w(t)=Ae^{-\omega_n t/2Q}\begin{pmatrix}\cos(-\omega_n t+\varphi_A)\\ \sin(-\omega_n t+\varphi_A)\end{pmatrix}+Be^{\omega_n t/2Q}\begin{pmatrix}\cos(\omega_n t+\varphi_B)\\ \sin(\omega_n t+\varphi_B)\end{pmatrix} \tag{1.65}$$

---

[23] We want to solve the general equation: $\ddot{z}'+\frac{\omega_n^2}{Q\omega_s}\dot{z}'+\left(\omega_n^2-j\frac{\omega_n^2}{Q}\right)z'=f(t)$

[24] We Start from equations (1.20) and (1.21) for the Jeffcott rotor. If $c_{NR}=0$ the eigenvalues are:

$$\lambda_{1,2}=\mp\frac{1}{\sqrt{2}}\sqrt{\omega_n^2+\sqrt{\omega_n^4+\left(\frac{k}{m_r Q}\right)^2}}+j\left[\frac{k/m_r}{2Q\omega_s}\pm\frac{1}{\sqrt{2}}\sqrt{-\omega_n^2+\sqrt{\omega_n^4+\left(\frac{k}{m_r Q}\right)^2}}\right]$$

Remembering the definition of the natural frequency ω_n, it follows:

$$\lambda_{1,2}=\mp\frac{1}{\sqrt{2}}\sqrt{\omega_n^2+\sqrt{\omega_n^4\left(1+\frac{1}{Q^2}\right)}}+j\left[\frac{\omega_n^2}{2Q\omega_s}\pm\frac{1}{\sqrt{2}}\sqrt{-\omega_n^2+\sqrt{\omega_n^4\left(1+\frac{1}{Q^2}\right)}}\right]$$

In the limit x>>1, we have $\sqrt{\left(1+1/x^2\right)}\approx 1+1/(2x^2)$.

Then, the following approximated relations for the eigenvalues are obtained:

$$\lambda_{1,2}\approx\mp\frac{1}{\sqrt{2}}\sqrt{2\omega_n^2\left(1+\frac{1}{4Q^2}\right)}+j\left[\frac{\omega_n^2}{2Q\omega_s}\pm\sqrt{\frac{\omega_n^2}{2Q^2}}\right]\approx\mp\omega_n\left(1+\frac{1}{8Q^2}\right)+j\left[\frac{\omega_n^2}{2Q\omega_s}\pm\frac{\omega_n}{2Q}\right]\approx\mp\omega_n\pm j\frac{\omega_n}{2Q}$$

$\vec{\xi}_w(t)$ in (1.65) is the so called whirl motion. It is the superimposition of a circular forward whirl motion (i.e. occurring in the same direction of the spin speed) which is self-excited, and a circular backward whirl motion which is damped. They both occur at an angular velocity equal to the natural frequency of the non-rotating system.
The solution of equation (1.59) can be obtained by adding the general solution of the homogeneous equation (1.65) to the particular integral (1.63) of the complete equation:

$$\begin{aligned}\vec{\xi} &= \vec{\xi}_\varepsilon(t) + \vec{\xi}_w(t) \\ &= \frac{\omega_n^2}{\left(\omega_n^2 - \omega_s^2\right)}\vec{\varepsilon}(t) + Ae^{-\omega_n t/2Q}\begin{pmatrix}\cos(-\omega_n t + \varphi_A) \\ \sin(-\omega_n t + \varphi_A)\end{pmatrix} + Be^{\omega_n t/2Q}\begin{pmatrix}\cos(\omega_n t + \varphi_B) \\ \sin(\omega_n t + \varphi_B)\end{pmatrix} \quad (1.66)\end{aligned}$$

Clearly, the damping time constant of the system due to the dissipation is proportional to the quality factor Q, namely $\tau = 2Q/\omega_n$. If Q is large, whirl growth is very slow. Equation (1.59) has been integrated numerically and the results are shown in figures 1.9, 1.10 and 1.11.

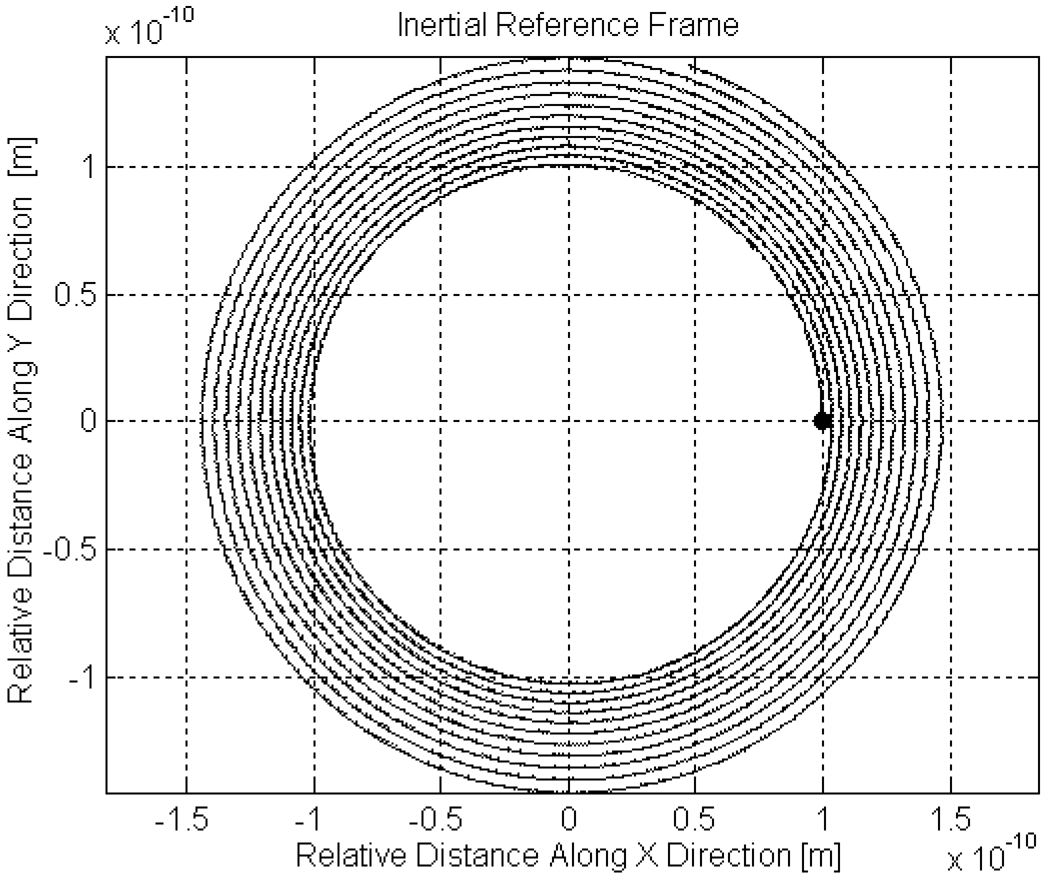

Figure 1.9: Simulation of the two body system.A forward whirl growing in amplitude. For demonstration purposes the numerical integration is carried with a bad quality factor (Q=10) to have a short time constant. Black circle: starting point.

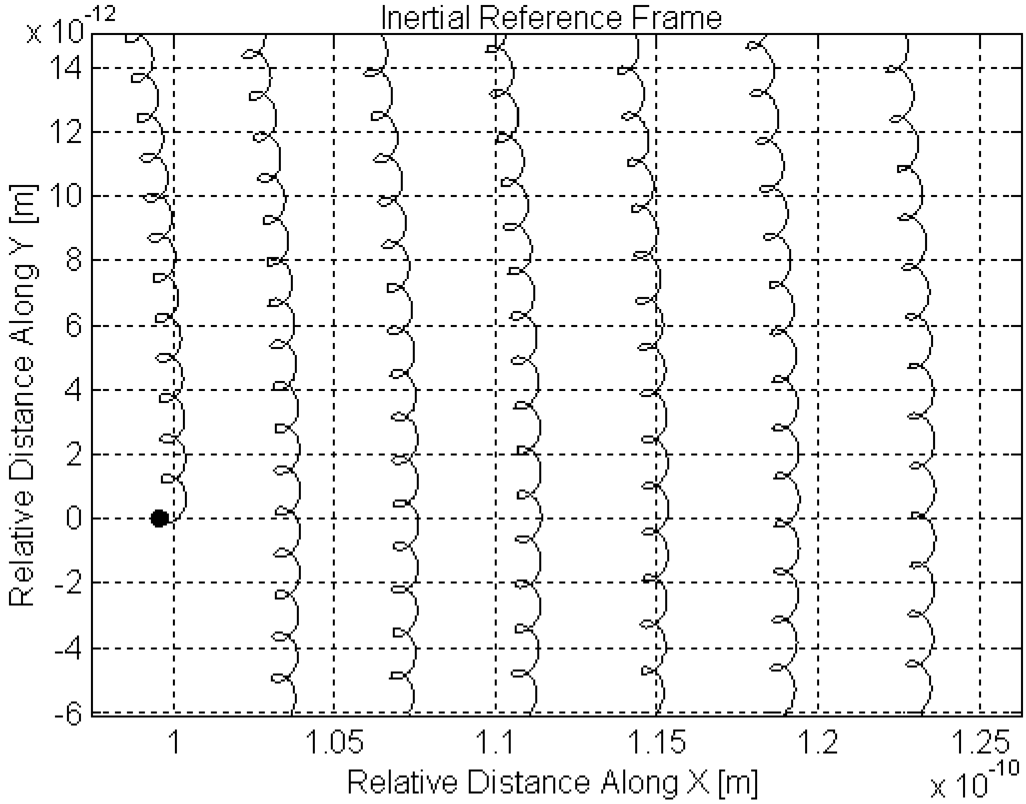

Figure 1.10: Enlargement of the previous figure. Black circle: starting point. Small oscillations at the spin frequency are visible.

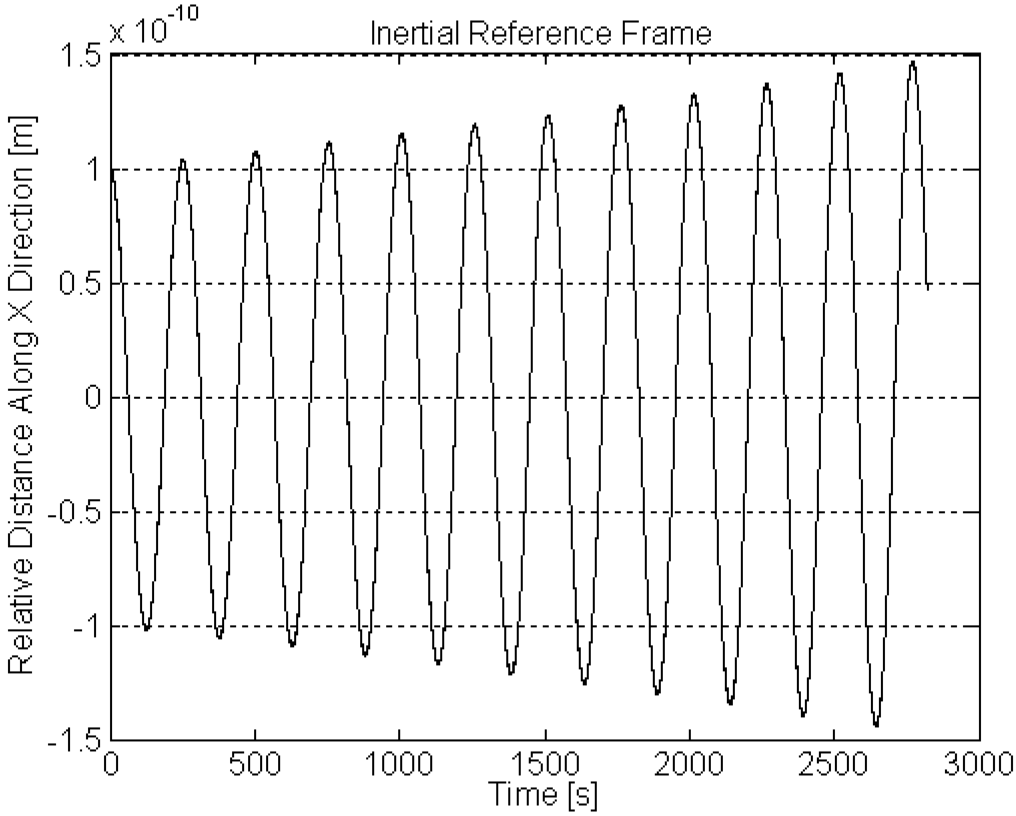

Figure 1.11: Simulation of the two body system. Relative distance along y direction as a function of time. The growth of the amplitude is evident.

We can now complete the mathematical model specializing here to the introduction of an external force F_e (constant or slowly variable) acting on body 1; the solution is the superimposition of three relative displacements:

$$\vec{\xi}=\vec{\xi}_{\varepsilon}(t)+\vec{\xi}_{w}(t)+\vec{\xi}_{Fe}(t) \qquad (1.67)$$

with:

$$\vec{\xi}_{Fe}(t)=-\frac{m_1}{k/m_r+\omega_s^2c_r^2/km_r}\vec{F}_e(t)-\frac{m_1}{k/m_r+\omega_s^2c_r^2/km_r}\frac{1}{Q}\hat{z}\times\vec{F}_e(t) \qquad (1.68)$$

$\vec{\xi}_{Fe}(t)$ in (1.68) is the vector describing the displacement of the equilibrium position due to the action of the external force. This result is interesting. For example, even though the external force has been applied along the x direction, finite differential displacements occurs along the y direction, due to the rotation and to the dissipative nature of the suspensions (the quality factor Q is finite. See figure 1.12).

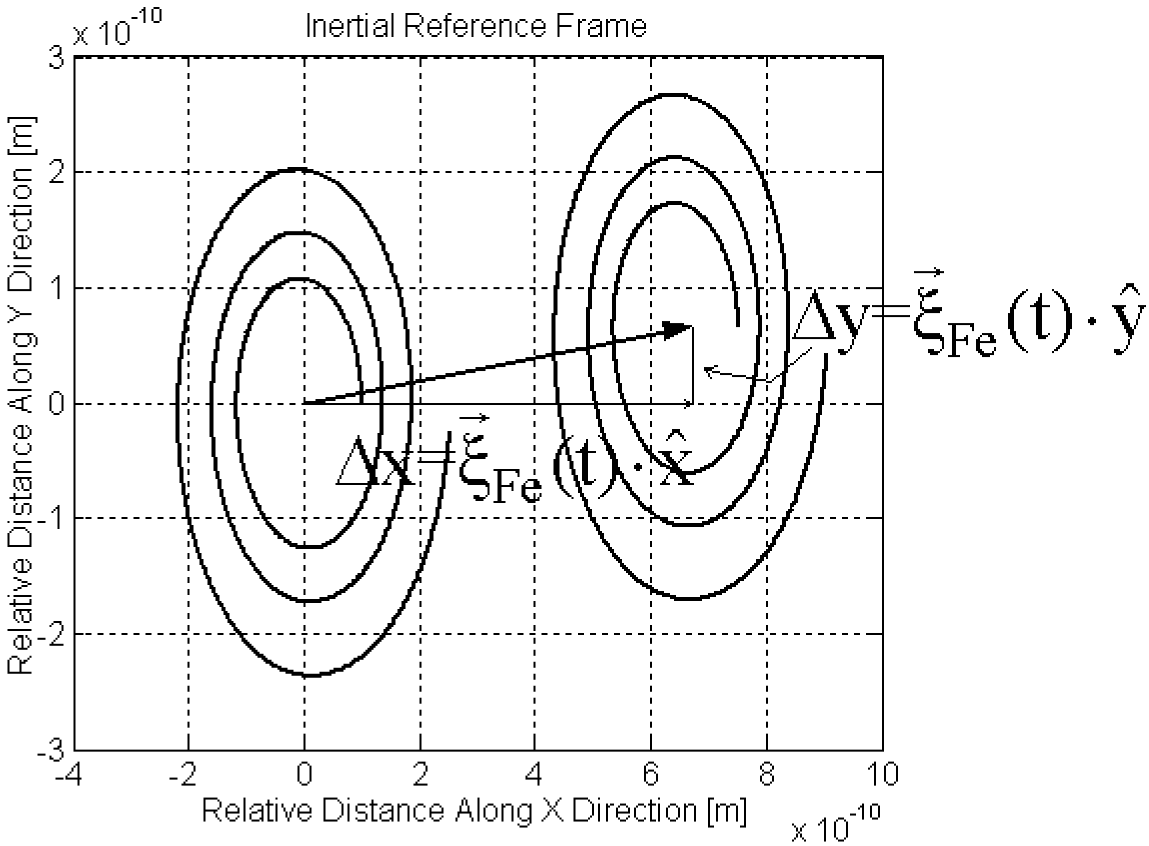

Figure 1.12: Simulation of the two body system. Relative motion in the inertial reference frame. The figure shows the plane of motion in two cases: with (right) and without (left) the inertial force acting on body 2. It shows how the effect of an external force is that of displacing the equilibrium position of the system. A small phase lag (Δy/Δx) appears due to energy losses in the suspensions (i.e. finite quality factor Q=10). Due to these losses, whirl motion at the natural frequency of the system arises in either case around the corresponding equilibrium position.

However, the magnitude of the displacement along the y direction ("orthogonal") is depressed by a factor $1/Q$ <<1 with respect to that along x. Hence, the two degrees of freedom are coupled from the energy dissipation only because Q has a finite value. The 2 body system is unstable. The simplest way to stabilize it is adding some non-rotating damping which is mathematically expressed by the terms containing the non-rotating damping coefficient c_{NR}. To this aim, we introduce the force (1.12) and then recast this equation in the rotating reference frame:

$$\ddot{\vec{\zeta}}=-\omega_n^2\left(\vec{\zeta}-\vec{\varepsilon}\right)-\frac{c_{NR}}{m_r}\left(\dot{\vec{\zeta}}+\vec{\omega}_s\times\vec{\zeta}\right)-\frac{\omega_n^2}{\omega_s Q}\dot{\vec{\zeta}}+\omega_s^2\vec{\zeta}-2\vec{\omega}_s\times\dot{\vec{\zeta}} \quad (1.69)$$

$\vec{\zeta}$ is the relative displacement between the bodies in the rotating reference frame. In agreement with (1.23), the condition for stability is $c_{NR}=c_R\ \omega_s/\omega_n$. Equation (1.69) has been integrated numerically and the result is shown in figure 1.13. This plot shows the approaching of the relative displacement to the equilibrium position.

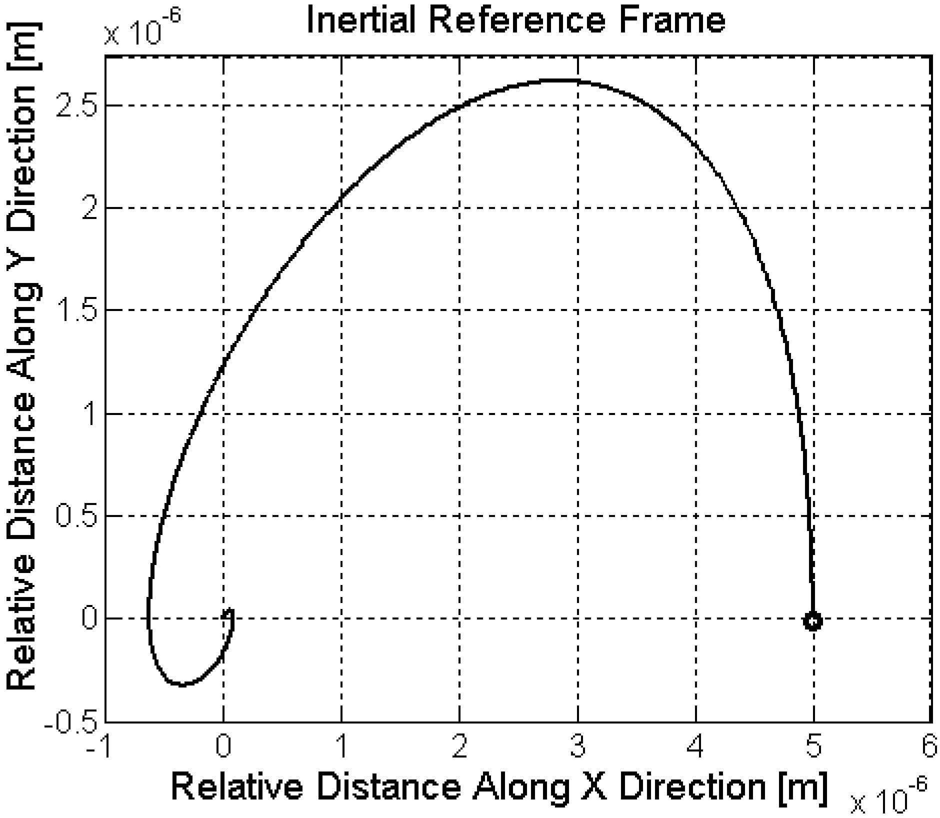

Figure 1.13: Simulation of the two body system. Polar plot of the relative displacement in the inertial reference frame. Forward whirl is damped. $c_{NR}=100\ c_R\ \omega_s/\omega_n$.

1.8: WHIRLING MOTION AND STABILIZING FORCE.

If there is friction inside rotating parts of the system this amounts to a non zero coefficient of rotating damping. Rotating damping has a destabilizing effect because it produces a spin down of the system and a corresponding forward whirling motion of the rotating bodies with an exponentially increasing amplitude.

In the inertial frame, the frequency of the whirling motions is essentially the natural frequency of the non-rotating system $\omega_w=\omega_n$ (ω_n is the natural frequency of the system). In the rotating frame, the deformations of the springs occur at the frequency $\omega_s \pm \omega_w$. As a consequence, whirl motions grows in amplitude at a rate which depends on the *Q* of the system at the frequency $\omega_s \pm \omega_w$, which is essentially the spin frequency: the higher the *Q* at this frequency, the slower the growth rate of the whirl. More precisely, rotor dynamics predicts that whirl grows with a negative *Q* opposite to the *Q* of the system at its spin frequency. This means that if r_w is the amplitude of the whirling motion, it increases in time as follows:

$$r_w(t)=r_w(0)e^{\frac{\omega_n t}{2Q}} \tag{1.70}$$

and its relative variation in one natural period T_n (if Q >>1) is:

$$\frac{\Delta r_w}{r_w(0)} \cong \frac{\omega_n}{2Q} T_n \tag{1.71}$$

This increase in the amplitude can be interpreted as due to an increasing of the along track velocity, caused by a destabilizing acceleration a_d such that:

$$\frac{1}{2} a_d T_n^2 \cong 2\pi \Delta r_w \tag{1.72}$$

or

$$a_d \cong \frac{1}{Q} \omega_n^2 r_w(0) \tag{1.73}$$

The destabilizing force [12] connected to the acceleration a_d can be written as:

$$\left|\vec{F}_d\right| = \frac{1}{Q} m\omega_n^2 r_w(0) = \frac{1}{Q}\left|\vec{F}_{centrifugal}\right| = \frac{1}{Q}\left|\vec{F}_{spring}\right| \tag{1.74}$$

where $\vec{F}_{centrifugal}$ is the centrifugal force and $\vec{F}_{spring}$ the elastic force of the spring. The destabilizing force is, then, a small fraction $1/Q<<1$ of the elastic force. An active damping force opposite to $\vec{F}_d$ (1.74) and slightly larger is required to stabilize the system.

1.9: ENERGY DISSIPATION IN WHIRLING MOTION.

At this point we want to evaluate how much energy is gained by the whirling motion as fraction of the energy lost by the spinning rotor [13]. The spin energy of the rotor is:

$$E_{rotor} = \frac{1}{2} I \omega_s^2 \tag{1.75.a}$$

with I the moment of inertia. The along track velocity is $v_w = \omega_n r_w$ and the energy of whirl motion is:

$$E_w = E_{kinetic} + E_{elastic} = \frac{1}{2} m\omega_n^2 r_w^2 + \frac{1}{2}\frac{k}{m} m r_w^2 = m\omega_n^2 r_w^2 \tag{1.75.b}$$

The time derivatives of the two energies (1.75.a) and (1.75.b) are:

$$\dot{E}_{rotor}=I\omega_s\dot{\omega}_s \quad ; \quad \dot{E}_w=2m\omega_w^2 r_w\dot{r}_w \tag{1.76}$$

Since the total angular momentum (the spin angular momentum of the rotor L_{rotor} plus the angular momentum of the whirl motion L_w) has to be conserved it must be:

$$L_{rotor}=I\omega_s; \quad L_w=m\omega_n r_w^2; \quad \dot{L}_w+\dot{L}_{rotor}=0 \tag{1.77}$$

Since the frequency ω_n is constant, it follows:

$$I\dot{\omega}_s+2m\omega_n r_w\dot{r}_w=0 \tag{1.78}$$

from which the derivative of the spin speed $\dot{\omega}_s$ can be obtained. By combining (1.77) with (1.78) we obtain:

$$\dot{E}_w=-\frac{\omega_n}{\omega_s}\dot{E}_{rotor} \tag{1.79}$$

Equation (1.79) shows that, in highly supercritical regime ($\omega_s>>\omega_n$), the energy gained by the whirling motion is a very small fraction of the energy lost by the rotor. All the rest is dissipated as heat inside the springs.

1.10: ISOTROPIC JEFFCOTT ROTOR ON NON-ISOTROPIC SUPPORTS.

In the study of rotating machinery a common assumption is that of axial symmetry of the rotor. If both stator and rotor are isotropic with respect to the rotation axis, particularly simple models can be built. On the contrary, if the rotor cannot be considered axially symmetrical, the study can become very complicated. In the following, we will study the simple model of the non-isotropic Jeffcott rotor [1]. With respect to the model in section 1.2, the only difference is, now, that the stiffness of the supports is not isotropic in the x-y plane. Assuming that the elastic constant along the x direction $k_x=k$ is lower than that along the y direction k_y, we introduce the non-dimensional parameter $\Lambda>1$ so that $k_y=\Lambda k_x$ and $k_x=k$.
The Lagrangean of the system is expressed by the relation:

$$\mathcal{L}=\frac{1}{2}m\left[\dot{x}^2+\dot{y}^2+\varepsilon^2\omega_s^2+2\varepsilon\omega_s\left(\dot{y}\cos(\omega_s t)-\dot{x}\sin(\omega_s t)\right)\right]-\frac{1}{2}k\left(x^2+\Lambda^2 y^2\right) \tag{1.80}$$

By performing the relevant derivatives , the following equations of motion are obtained:

$$\begin{cases} m\left[\ddot{x}-\varepsilon\omega_s^2\cos(\omega_s t)\right]+kx=0 \\ m\left[\ddot{y}-\varepsilon\omega_s^2\sin(\omega_s t)\right]+\Lambda ky=0 \end{cases} \tag{1.81}$$

The homogeneous equations of motion associated with the system (1.81) are coincident with the equations of the free motion of two system with one degree of freedom and their solutions are two harmonic motions with frequencies:

$$\omega_x=\sqrt{k/m} \quad ; \quad \omega_y=\sqrt{\Lambda k/m}=\sqrt{\Lambda}\omega_x \tag{1.82}$$

The two natural frequencies are independent from the spin speed and coincide with the two critical speeds of the system. The particular solution related to the presence of the unbalance is readily obtained imposing $\ddot{x}=0$ and $\ddot{y}=0$.

$$\begin{bmatrix} x_p(t) \\ y_p(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\varepsilon\omega_s^2}{\omega_x^2-\omega_s^2}\cos(\omega_s t) \\ \dfrac{\varepsilon\omega_s^2}{\omega_y^2-\omega_s^2}\sin(\omega_s t) \end{bmatrix} \tag{1.83}$$

Starting from the definition of the centre of mass $\vec{r}_G$, it follows:

$$\begin{bmatrix} x_{Gp}(t) \\ y_{Gp}(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\varepsilon\omega_x^2}{\omega_x^2-\omega_s^2}\cos(\omega_s t) \\ \dfrac{\varepsilon\omega_y^2}{\omega_y^2-\omega_s^2}\sin(\omega_s t) \end{bmatrix} = \varepsilon \begin{bmatrix} \dfrac{1}{1-\omega_s^2/\omega_x^2}\cos(\omega_s t) \\ \dfrac{1}{\Lambda}\dfrac{1}{1-\omega_s^2/\left(\Lambda\omega_x^2\right)}\sin(\omega_s t) \end{bmatrix} \tag{1.84}$$

Let us now introduce the complex coordinate $z'=x+jy$ and define the elastic constants k_m and k_d:

$$k_m = \left(k_x + k_y\right)/2 \tag{1.85.a}$$

$$k_d = \left(k_x - k_y\right)/2 = k_x(1-\Lambda)/2 \tag{1.85.b}$$

the particular solution can be written in the form:

$$z'_p(t) = \frac{\varepsilon m\omega_s^2}{\left(k-m\omega_s^2\right)\left(\Lambda k - m\omega_s^2\right)}\left\{\left[k_m - m\omega_s^2\right]e^{j\omega_s t} - k_d e^{-j\omega_s t}\right\}^{25} \tag{1.86}$$

When the equality $k_m - m\omega_s^2 = 0$ is satisfied (i.e. $\omega_s = \sqrt{k_m/m}$), the amplitude of the forward whirl vanishes and the motion is a circular backward whirl with amplitude $\varepsilon k_m/k_d$. This is an important result: in presence of anisotropy, backward whirling motions can be self-excited. The amplitude of motion of the point mass G and the components of the vector $\vec{r}_G$ are reported as a function of the spin speed in the non-dimensional plot of figure 1.14. This figure shows the presence of three different speed ranges:

- in the range from 0Hz to the first critical speed (1.82.a) the components of the vector $\vec{r}_G$ are positive and they are out of phase from each other by 90° (see equation (1.84)

[25] Starting from (1.81) $z'_p = x_p + jy_p = \varepsilon\omega_s^2\left[\cos(\omega_s t)/\left(\omega_x^2-\omega_s^2\right) + j\sin(\omega_s t)/\left(\omega_y^2-\omega_s^2\right)\right]$. Thanks to the well known formula $\cos(\omega_s t) = \left(e^{j\omega_s t}+e^{-j\omega_s t}\right)/2$ and $\sin(\omega_s t) = \left(e^{j\omega_s t}-e^{-j\omega_s t}\right)/(2j)$, it follows:

$$z'_p = \varepsilon\omega_s^2\left\{\left[(\omega_x^2+\omega_y^2)/2-\omega_s^2\right]e^{j\omega_s t} + (\omega_y^2-\omega_x^2)e^{-j\omega_s t}/2\right\}\Big/\left[\left(\omega_x^2-\omega_s^2\right)\left(\omega_y^2-\omega_s^2\right)\right].$$

By replacing $\omega_x = \sqrt{k/m}$ and $\omega_y = \sqrt{\Lambda k/m} = \sqrt{\Lambda}\omega_x$, equation (1.86) is easily obtained.

or equation (1.86)). The orbit in the x-y plane is elliptic. The x component grows from ε to a value tending to infinity at the first critical speed, while the y component has a finite value. Hence, in the limit $\omega_s \to \omega_x$, the axis of orbit along the x direction tends to infinity.

- In the range from the first to the second critical speed, x_G is negative and y_G is positive. At the frequency $\omega_s = \sqrt{k_m / m}$ (see equation (1.86)) the amplitude of the forward whirl vanishes and the motion is a circular backward whirl with amplitude $\varepsilon k_m / k_d$. In the range from the first critical speed to $\omega_s = \sqrt{k_m / m}$, the ellipse is much elongated along the x direction. In the range from $\omega_s = \sqrt{k_m / m}$ to the second critical speed the ellipse is much elongated along the y direction. In the limit $\omega_s \to \omega_y$, the axis of orbit along the y direction tends to infinity.

- In the supercritical region ($\omega_s > \omega_y$), the components of the vector $\vec{r}_G$ are negative and they tend to zero when the spin speed tends to infinity. Hence, in supercritical region there is a self-centring of the body on the rotation axis and the elastic anisotropy has negligible effect on the rotor.

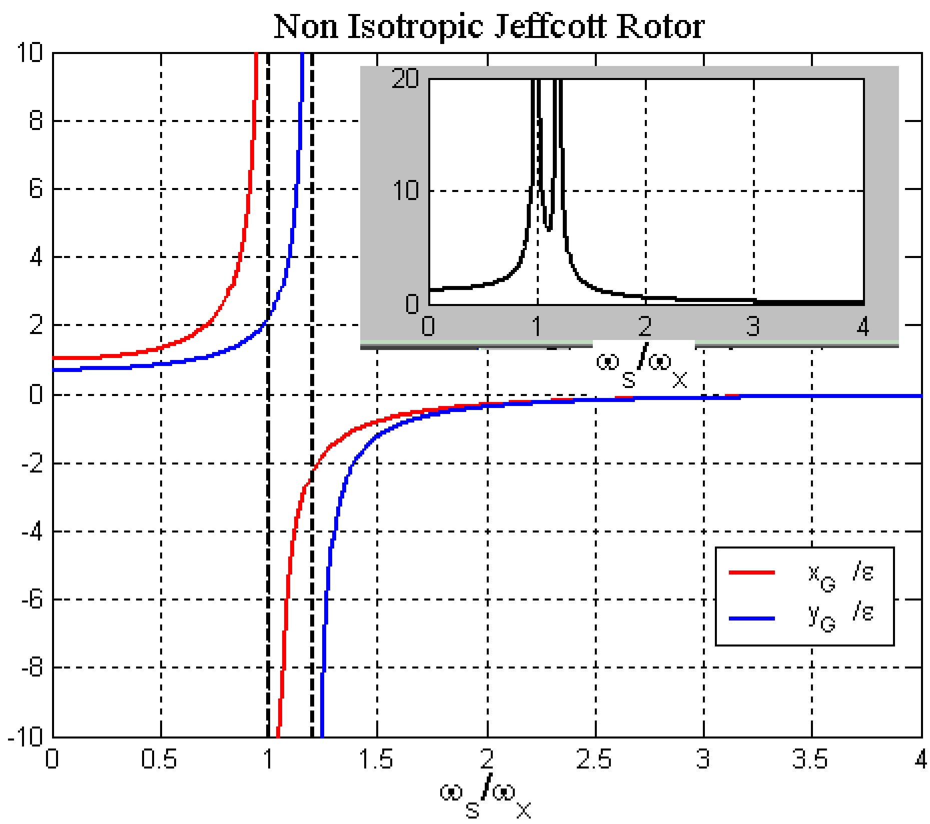

Figure 1.14:Non dimensional response of the Jeffcott Rotor on non-isotropic supports. Red curve: x_G /ε. Blue curve: y_G /ε. Inset: amplitude of the vector $\vec{r}_G$. Λ=1.5.

1.11: NON-ISOTROPIC JEFFCOTT ROTOR. NATURAL FREQUENCIES.

In this section, we will consider a Jeffcott rotor in which the shaft is not isotropic (the elastic constants along the direction ξ and η of the rotating frame are k_ξ and $k_\eta = \Lambda' k_\xi$) . Since the deviation from symmetry concern the rotor (not the stator), better evidence can be obtained by writing the equation of motion with reference to the rotating frame (O,ξ,η,z).
Starting from the Lagrange function, the equations of motion[26] can be written in a compact matrix form as:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{\xi} \\ \ddot{\eta} \end{bmatrix} + \begin{bmatrix} 0 & -2m\omega_s \\ 2m\omega_s & 0 \end{bmatrix}\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} k_\xi - m\omega_s^2 & 0 \\ 0 & k_\eta - m\omega_s^2 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = m\varepsilon\omega_s^2 \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix} \tag{1.87}$$

After performing the relevant derivatives with the assumptions $\ddot{\xi} = \ddot{\eta} = \dot{\xi} = \dot{\eta} = 0$, the (1.87) yields two homogeneous algebraic equations:

$$\begin{bmatrix} k_\xi - m\omega_s^2 & 0 \\ 0 & k_\eta - m\omega_s^2 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = m\varepsilon\omega_s^2 \begin{bmatrix} \cos(\alpha) \\ \sin(\alpha) \end{bmatrix} \tag{1.88}$$

The equilibrium position in the rotating frame is easily obtained:

$$\begin{bmatrix} \xi_{eq} \\ \eta_{eq} \end{bmatrix} = \begin{bmatrix} \dfrac{m\varepsilon\omega_s^2 \cos(\alpha)}{k_\xi - m\omega_s^2} \\ \dfrac{m\varepsilon\omega_s^2 \sin(\alpha)}{k_\eta - m\omega_s^2} \end{bmatrix} \tag{1.89}$$

The critical speeds are the frequencies at which the two denominators in (1.89) vanish:

[26] Let us start by defining the relevant quantities for this problem; the eccentricity is $\vec{\varepsilon} = \varepsilon(\cos(\alpha), \sin(\alpha))$ where α is the angle between the ξ axis and the direction of the vector. The position vector of the point mass G is $\vec{r}_G = \vec{\varepsilon} + (\xi, \eta, 0)$. The spin angular velocity is $\vec{\omega}_s = \omega_s(0,0,1)$, i.e. it is aligned with the vertical axis z. We have to write the potential and kinetic energies in order to write the lagrangean function of the system: the kinetic energy is $T = m\dot{\vec{r}}_G^2/2 = m(\dot{\xi}^2 + \dot{\eta}^2)/2$ while the centrifugal potential energy can be written in the form: $U_c = m(\vec{\omega}_s \times \vec{r}_G)^2/2 = m\omega_s^2(\xi^2 + \eta^2 + 2\varepsilon\xi\cos(\alpha) + 2\varepsilon\eta\sin(\alpha))/2$ (see appendix 3.A for the definition of the "centrifugal" term of the lagrangean function of a body spinning in a rotating reference frame; note that $\vec{\varepsilon}$ is constant in the rotating frame, hence the term $m\omega_s^2\varepsilon^2/2$, can be neglected); the "Coriolis" potential energy $U_{cc} = m\dot{\vec{r}}_G \cdot (\vec{\omega}_s \times \vec{r}_G) = -m\omega_s(\dot{\xi}\eta - \dot{\eta}\xi)$ (see appendix 3.A for the definition of the "Coriolis" term of the lagrangian function of the system); the elastic potential energy $U_k = k_\xi\xi^2/2 + k_\eta\eta^2/2$. The operative expression for the Lagrange's function in the rotating frame is then: $\mathcal{L} = T + U_c + U_{cc} - U_k$.

$$\omega_\xi = \sqrt{\frac{k_\xi}{m}} \tag{1.90.a}$$

$$\omega_\eta = \sqrt{\frac{k_\eta}{m}} = \sqrt{\Lambda'}\omega_\xi \tag{1.90.b}$$

The characteristic equation associated with equation (1.87) is:

$$\left\| \begin{bmatrix} -\omega^2 + \omega_\xi^2 - \omega_s^2 & -2j\omega_s\omega \\ 2j\omega_s\omega & -\omega^2 + \omega_\eta^2 - \omega_s^2 \end{bmatrix} \right\| = 0 \tag{1.91}$$

By introducing the non-dimensional parameters:

$$\lambda_\xi = \frac{\omega}{\omega_\xi},\ \omega' = \frac{\omega_s}{\omega_\xi} \tag{1.92}$$ [27]

equation (1.91) can be written in the form:

$$\lambda_\xi^4 - \lambda_\xi^2\left[\Lambda' + 2\omega'^2 + 1\right] + \left(1 - \omega'^2\right)\left(\Lambda' - \omega'^2\right) = 0 \tag{1.93}$$

By solving equation (1.93) in λ_ξ^2 it follows:

$$\lambda_\xi^2 = \omega'^2 + \frac{1+\Lambda'}{2} \pm \sqrt{2\omega'^2\left(1+\Lambda'\right) + \frac{\left(\Lambda'-1\right)^2}{4}} \tag{1.94}$$

The expression under the radical sign $\Delta^2 = 2\omega'^2\left(1+\Lambda'\right) + \left(\Lambda'-1\right)^2/4$ is always positive and the solutions λ_ξ^2 of (1.94) are always real. The one with the sign + is positive, hence there are two real solutions in λ_ξ:

$$\lambda_{\xi,1,2} = \pm\sqrt{\omega'^2 + \frac{1+\Lambda'}{2} + \Delta} \tag{1.95}$$

The root with sign – is positive only if $\omega'^2 + \left(1+\Lambda'\right)/2 - \Delta > 0$; after some simple algebra, this condition can be written as:

$$\omega'^4 - \omega'^2\left(1+\Lambda'\right) + \Lambda' > 0 \tag{1.96}$$

Let us define the function $f(\omega') = \omega'^4 - \omega'^2\left(1+\Lambda'\right) + \Lambda'$. It is easy to show that it can be written as the product of two polynomials of second order in ω':

$$f(\omega') = \left(\omega'^2 - 1\right)\left(\omega'^2 - \Lambda'\right) \tag{1.97}$$

If $f(\omega')$ is positive, the characteristic equation (1.93) has 4 real roots and the system is stable. If $f(\omega')$ is negative (i.e. when $1 < |\omega'| < \sqrt{\Lambda'}$), the characteristic equation (1.93) has 2 real and 2 complex roots. One of the two complex roots has a negative imaginary part which corresponds to an unstable behaviour of the system.

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[27] $\omega'=1$ means that the system rotates at angular speed $\omega_s=\omega_\xi$. $\omega'=\sqrt{\Lambda'}$ means that the system rotates at angular speed $\omega_s=\omega_\eta$. In the same manner: $\lambda_\xi=1 \Rightarrow \omega=\omega_\xi$ and $\lambda_\xi=\sqrt{\Lambda'} \Rightarrow \omega=\omega_\eta$.

$f(\omega') > 0$	$\lvert\omega'\rvert < 1 \vee \lvert\omega'\rvert > \sqrt{\Lambda'}$	4 Real Roots	Stable
$f(\omega') < 0$	$1 < \lvert\omega'\rvert < \sqrt{\Lambda'}$	2 Real Roots 2 Complex Roots	Unstable

Table 1.1:Roots of the characteristic equation (1.93).

The presence of the anisotropy causes the occurrence of an instability range that spans from ω_ξ to ω_η[28]. Rotating and non-rotating damping reduce the instability range between the two critical speeds. In the inertial frame we can introduce the non-dimensional parameter:

$$\lambda^{NR} = \omega^{NR} / \omega_\xi \tag{1.98}$$

that is the non dimensional whirl speed in the x-y plane. λ^{NR} is linked with λ_ξ by the relationship:

$$\lambda^{NR} = \lambda_\xi + \omega' \tag{1.99}$$

By combining equation (1.94) with (1.99), the whirl frequencies in the inertial frame are easily obtained:

$$\lambda^{NR} = \omega' \pm \sqrt{\omega'^2 + \frac{1+\Lambda'}{2} \pm \sqrt{2\omega'^2\left(1+\Lambda'\right) + \frac{\left(\Lambda'-1\right)^2}{4}}} \tag{1.100}$$

The dynamical behaviour of the system (Λ'=1.5) is summarized by figure 1.15 where the natural frequencies are shown as function of the non-dimensional spin speed ω' in the inertial reference frame. Only the first and the fourth quadrant are depicted, because they give a complete picture of the situation (the second and the third quadrant refer to the case of clockwise spin frequency). The frequency range between ω' =1 and $\omega' = \sqrt{\Lambda'}$ =1.22 is the instability range (we have seen that the system is unstable when (1.97) is negative, i.e. when $1 < \lvert\omega'\rvert < \sqrt{\Lambda'}$). The cyan dashed line $\lambda^{NR}=\omega'$ separates the supercritical ($\lambda<\omega'$) from the subcritical ($\lambda^{NR}>\omega'$) region. There are four natural frequencies (the four solutions of equation (1.93)) that form four branches (yellow, blue, violet, red lines).

For example, if $\omega'=\varpi$ the equation (1.93) has four roots (open circles 1,2,3,4 in figure): they are found by the intersection between the vertical black line and the 4 coloured branches (yellow, blue, violet, red lines). The root numbered as 1 in figure 1.15 corresponds to a backward whirling motion (the corresponding value of λ^{NR} is negative). The roots numbered as 2,3,4 correspond to forward whirling motions. In the high supercritical range (ω'>>1), two

[28] In section 1.2 we have stated that self-centring is possible only if the system has al least two degrees of freedom. 1D systems are highly unstable if spinning at frequencies higher than the natural one. This is a consequence of the previous result. In fact, a system with one degree of freedom can be considered a limiting case of asymmetrical rotor. The stiffness along the η axis can be considered infinitely high and the corresponding critical speed is infinitely high too. The instability range spans from ω_ξ to $\omega_\eta=\infty$, i.e. it extends for all values of spin speeds that are above the critical frequency ω_ξ.

natural frequencies grow linearly with $2\nu_s$, and the other two solutions are approximately constant[29]:

➢ $\omega' \to \infty$

$$\lambda^{NR} = \begin{cases} 2\omega' \pm \sqrt{(1+\Lambda')/2} \simeq 2\omega' \\ \pm\sqrt{(1+\Lambda')/2} = \pm\sqrt{k_m / k_\xi} \end{cases} \tag{1.101}$$

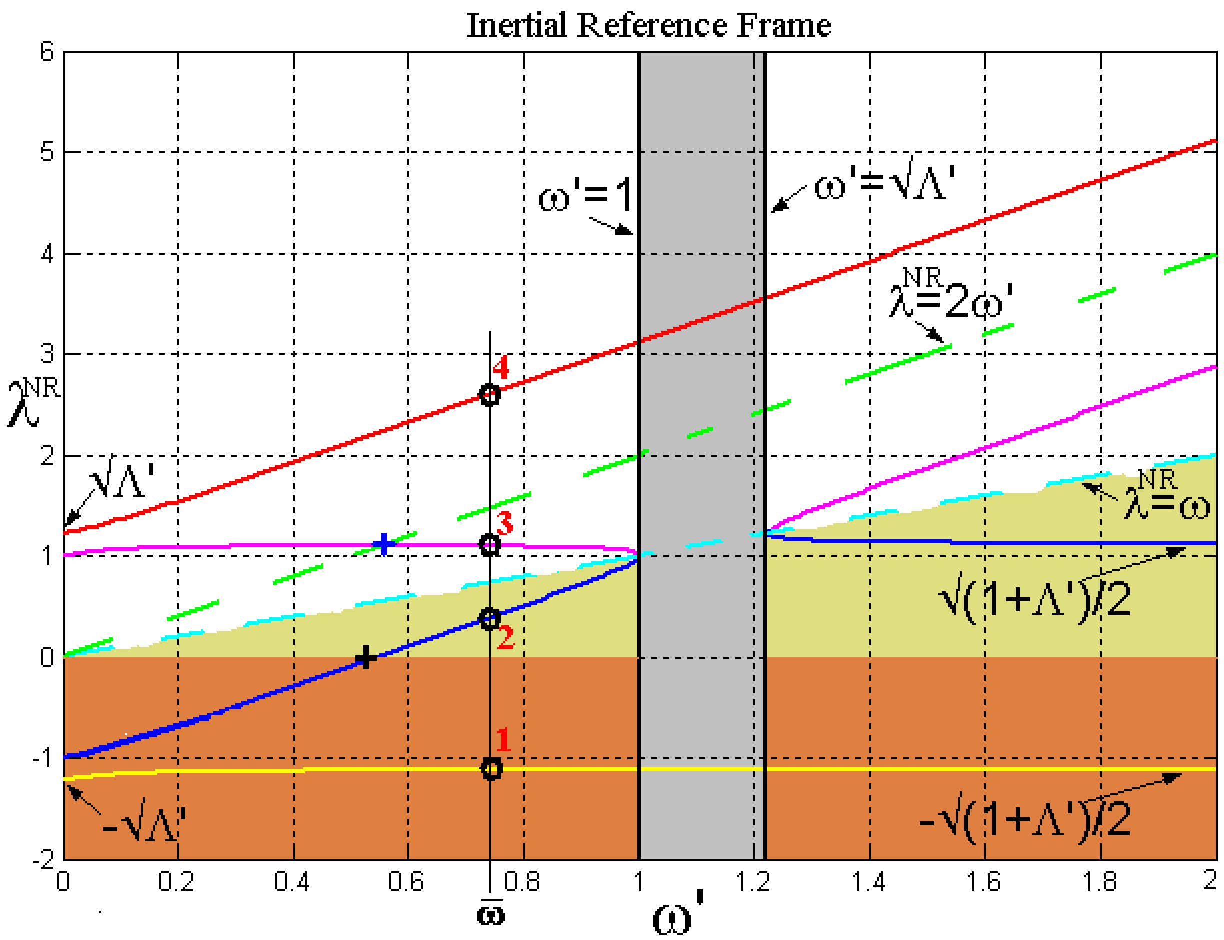

Figure 1.15: Non-isotropic (Λ'=1.5) Jeffcott rotor. Non-dimensional natural frequencies shown as function of the non-dimensional spin speed ω' in the inertial reference frame. The frequency range between $\omega'=1$ and $\omega'=\sqrt{\Lambda'}=1.22$ is the instability range. The cyan dashed line $\lambda^{NR}=\omega'$ separates the supercritical ($\lambda^{NR}<\omega'$) from the subcritical ($\lambda^{NR}>\omega'$) region. There are four natural frequencies (yellow, blue, violet, red lines) at each spin speed (the four solutions of equation (1.93)). For example, if $\omega'=\varpi$ the equation (1.93) has four roots (1,2,3,4): they are found by the intersection between the vertical black line and the 4 coloured branches (yellow, blue, violet, red lines). The root numbered as 1 corresponds to a backward whirling motion (the corresponding value of λ^{NR} is

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[29] In the limit $\omega' \to \infty$, from equation (1.100) we obtain $\lambda^{NR} \simeq \omega' \pm \sqrt{\omega'^2 \pm \omega'\sqrt{2(1+\Lambda')}}$. Sign + before the radical sign: $\lambda^{NR} \simeq \omega' + \sqrt{\omega'^2 \pm \omega'\sqrt{2(1+\Lambda')}} = \omega' + \omega'\left(1 \pm \sqrt{(1+\Lambda')/2}/\omega'\right) = 2\omega' \pm \sqrt{(1+\Lambda')/2}$. Sign - before the radical sign:

$$\lambda^{NR} \simeq \omega' - \sqrt{\omega'^2 \pm \omega'\sqrt{2(1+\Lambda')}} = \omega' - \omega'\sqrt{1 \pm \sqrt{2(1+\Lambda')}/\omega'} = \omega' - \omega'\left(1 \pm \sqrt{(1+\Lambda')/2}/\omega'\right) = \pm\sqrt{(1+\Lambda')/2}$$

negative). The root numbered as 2 corresponds to a forward whirling motion. Roots numbered as 3 and 4 are subcritical (forward). In the high supercritical range ($\omega' >> 1$) two natural frequencies grow linearly with $2\nu_s$, and the other two solutions are approximately constant ($\pm\sqrt{(1+\Lambda')/2}$). In the low frequency regime ($\omega' << 1$) the natural frequencies are coincident with the critical frequencies. Blue cross: the second critical speed ω_{cr2} located at the intersection of the violet branch with the $\lambda=2\omega'$ axis. Black cross: the intersection of the blue branch with the $\lambda=0$ axis.

where we have used the mean constant $k_m = (k_\xi + k_\eta)/2$. In the low frequency range ($\omega' << 1$), the natural frequencies are coincident with the critical frequencies[30]:

- $\omega'=0$ $$\lambda^{NR} = \begin{cases} \pm\sqrt{\Lambda'} \\ \pm 1 \end{cases} \quad (1.102)$$

In figure 1.15 a second critical spin speed ω'_{cr2} is shown at the intersection of the free whirling violet branch with the straight line $\lambda^{NR}=2\omega'$. All second critical frequency occur in the subcritical region and can not produce unstable whirl. Figure 1.15 also shows an intersection between the blue branch and the $\lambda^{NR}=0$ axis. The value of the spin frequency at which this intersection occur is[31]:

$$\omega'_{\lambda^{NR}=0} = \sqrt{\frac{1}{2}\frac{\Lambda'}{(1+\Lambda')}} \quad (1.103)$$

This frequency is about half of the primary critical speed ω_ξ[32]:

$$\omega_{\lambda^{NR}=0} = \omega'_{\lambda^{NR}=0} \cdot \omega_\xi = \sqrt{\frac{1}{2m}\frac{k_\xi k_\eta}{k_\xi + k_\eta}} \approx \frac{\omega_\xi}{2} \quad (1.104)$$

There is then, at a well-determined spin speed, a natural frequency that vanishes ($\lambda^{NR} \to 0$). At this speed the system is resonant with a static force (DC force), i.e. with a force constant in modulus and direction.

1.12: CONCLUSIONS.

In this chapter we have introduced the Jeffcott rotor to model the dynamical behaviour of the rotors. This simple model allows an understanding of the most important phenomena typical of rotor dynamic. The self-centring in supercritical rotation has been described in section 1.2 (see figure 1.3): the system will spin at a frequency either below or above the natural one. From (1.10), it follows that in the first case the equilibrium position will be farther away from the spin axis than the original offset ε, while in the second case equilibrium will take place

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[30] By inserting $\omega'=0$ in equation (1.100), it follows $\lambda^{NR} = \pm\sqrt{1+\Lambda' \pm (\Lambda'-1)}/\sqrt{2}$.

[31] By inserting $\lambda = 0$ into (1.100) the value of the speed is obtained.

[32] $\omega_{\lambda^{NR}=0} = \omega'\omega_\xi = \sqrt{\frac{1}{2}\frac{\Lambda'}{(1+\Lambda')}\frac{k_\xi}{k_\xi}}\sqrt{\frac{k_\xi}{m}} = \sqrt{\frac{1}{2m}\frac{k_\xi k_\eta}{k_\xi + k_\eta}}$

closer than ε to the spin axis. The higher the spin speed, the lower the distance of the centre of mass of the suspended body from the undeformed rotation axis. In supercritical regime, the rotors tend to rotate about their centre of mass instead of their geometrical centre. The final motion of the centre of mass is the superimposition of a free whirl (circular, elliptic or linear) at frequency ω_{cr} (see equation (1.8)) and a circular motion with angular velocity ω_s (see equation (1.10)).
In section 1.4 we have shown that rotating damping has a destabilizing effect on the rotor. In presence of rotating damping, the amplitude of whirling motions changes in time with exponential law: backward whirls (1.20) are stable, with decreasing amplitude, while forward whirl can be either damped or self excited (1.21). The condition for stability is given by the inequality (1.22): if only rotating damping is present the motion is unstable in whole supercritical regime; instead, non-rotating damping has a stabilizing effect on the rotor. The rotating damping is the friction (viscous plus structural) between the rotating parts of the rotor. The corresponding losses produce the instabilities in weakly suspended rotors in supercritical regime. The non-rotating damping is the friction between two non-rotating parts of the stator. It is effective in damping transverse translational oscillations of the spin axis without slowing down its rotation. A third kind of damping is the friction in the bearings. This is a friction between the rotor and the stator, which is effective in slowing down the rotation but it is ineffective at damping whirling motion.
In section 1.7 we have studied the problem of two weakly coupled rotors: energy dissipation makes the spin rate to decrease, together with the spin angular momentum. Since the total angular momentum must be conserved, the bodies develop a whirl motion. In supercritical regime, the final motion is the superimposition of a circular forward whirl motion (i.e. occurring in the same direction of the spin speed) which is self-excited (1.65), a circular backward whirl motion (1.65) which is damped (they both occur at an angular velocity equal to the natural frequency of the non-rotating system) and a circular motion with amplitude decreasing with the spin speed (1.63).
In section 1.8 we have evaluated the destabilizing force connected to the energy dissipation in the suspensions (1.74), showing that it is a small fraction ($1/Q<<1$) of the elastic force of the springs. Note that the growth rate of whirls is determined by losses in the system, essentially in the mechanical suspensions as they undergo deformations at the frequency spin and the relevant Q is that measured at the spin speed.
In section 1.9 we have evaluated the fraction of the energy lost by the spinning rotor gained by the whirling motion, showing that, in highly supercritical regime ($\omega_s>>\omega_n$), almost all the energy is dissipated as heat inside the springs (1.80) and do not contribute to the growth of the whirl.
In sections 1.10 and 1.11 we have studied the problem of the non-isotropic Jeffcott rotor. Equation (1.86) shows that, in the case of non-isotropic support (non-isotropic stator), backward whirling motions can be self-excited.
In section 1.11 we have considered a Jeffcott rotor in which the shaft is not isotropic. In the case of the non-rotating system ((1.102), (1.90.a) and (1.90.b)), the natural frequency is expected to split up (the system has two critical speeds ω_ξ and ω_η). In the high supercritical range (1.101), two natural frequencies grows linearly with $2\nu_s$, and the other two solutions are approximately constant. The presence of this anisotropy causes the occurrence of an instability range that spans from the first critical speed ω_ξ to the second critical speed ω_η

(figure 1.15 and table 1.1) which can be reduced by introducing non-rotating damping. The system has, then, at a well-determined spin speed (1.104), a natural frequency that vanishes and then a resonance with a DC force is possible.