

$$\vec{F} = -G \frac{m_1 m_2}{r^2} \left(\frac{\vec{r}}{r} \right)$$

$$\vec{F} = m \vec{a} = \frac{d}{dt} (m \vec{v})$$

Rif. Inerziale
Spazio assoluto
→ Tempo assoluto

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\vec{O} = m_1 \vec{r}_1 + m_2 \vec{r}_2$$

$$\mu \ddot{\vec{r}} = -G \mu M \frac{\vec{r}}{r^2}$$

$$\vec{L} = \frac{m_1 m_2}{M}$$

memoria
ridotta

$$\vec{J} = \mu \vec{r} \times \dot{\vec{r}}$$

momento angolare
orbitale totale

$$\dot{\vec{M}} = \dot{\vec{J}} = \vec{0} \Rightarrow \vec{J} = \text{costante}$$

$$\vec{F} \propto \frac{\vec{r}}{r^3} \Rightarrow \text{moto piano}$$

2 equazioni

$$l = d$$

$$\hat{e}_x = (1, 0)$$

$$\hat{e}_y = (0, 1)$$

$$\hat{e}_x \cdot \hat{e}_x = 1$$

$$\hat{e}_y \cdot \hat{e}_y = 1$$

$$\hat{e}_r = (\cos \vartheta, \sin \vartheta)$$

$$\hat{e}_\vartheta = (-\sin \vartheta, \cos \vartheta)$$

$$\hat{e}_r \cdot \hat{e}_\vartheta = 0$$

$$\vec{r} = (r \cos \vartheta, r \sin \vartheta)$$

$$\vec{r} = r \hat{e}_r$$

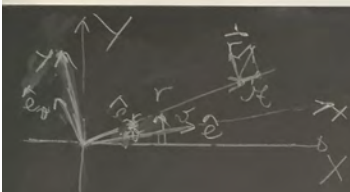
$$\dot{\vec{r}} = \dot{r} \hat{e}_r + r \dot{\hat{e}}_r = \dot{r} \hat{e}_r + r \dot{\vartheta} \hat{e}_\vartheta$$

$$\ddot{\vec{r}} =$$

- Problema ridotto, passaggio a coordinate polari piano e riduzione ad 1 sola variabile
- Integrale primo del vettore di Lenz e equazione dell'orbita

[illegible]

Problema 2 corpo ridotto ad 1 solo corpo $(\mathcal{H}, M)$ $\left[\begin{matrix} \vec{r}(t), \dot{\vec{r}}(t) \\ m_1, m_2 \end{matrix} \right]$ $\begin{matrix} Y \\ \uparrow \\ M \end{matrix}$ $\begin{matrix} \hat{e}_r, r \\ \nearrow \\ \mathcal{H} \end{matrix}$ $\begin{matrix} \hat{r} \\ \nearrow \\ \mathcal{H} \end{matrix}$ $X=0$
 $\downarrow$ $\boxed{\ddot{r} = \frac{r^2}{r^3} - \frac{GM}{r^2}}$ $\boxed{r(t)}$ 1 sola variabile, di distanza mutua, $r > 0$
 $(d = r^2 \dot{\varphi})$ integrale del moto $\mathcal{H} = m_1, m_2$ $M = m_1 + m_2$ $\boxed{\ddot{r} = -\frac{GM}{r^2}}$ $\begin{matrix} \hat{e}_z \\ \uparrow \\ \hat{e}_r \times \hat{e}_\varphi = -\hat{e}_\varphi \end{matrix}$
 $\vec{e} = \frac{1}{GM} \dot{\vec{r}} \times \vec{J} - \hat{e}_r$ Vettore di Lenz $\vec{J} = \vec{r} \times \vec{p}$ $\rightarrow$ grazie nel piano del moto.
 $[e] = \frac{m^2 s^{-2} m^2 s^{-1}}{m^2 s^{-2} m^2 s^{-1}}$ adimensionale
 $\vec{e} = \vec{0} \Rightarrow \vec{e}$ integrale del moto
 $\vec{r} \times \vec{J} = r \hat{e}_r \times (r^2 \dot{\varphi} \hat{e}_z) = r^3 \dot{\varphi} \hat{e}_r \times \hat{e}_z = -r^3 \dot{\varphi} \hat{e}_\varphi$
 $GM \vec{e} = \dot{\vec{r}} \times \vec{J} - GM \vec{e} r = \dot{\vec{e}} r$
 $GM \ddot{\vec{e}} = \ddot{\vec{r}} \times \vec{J} + \vec{r} \times \ddot{\vec{J}} - GM \ddot{\vec{e}} r$
 $\ddot{\vec{e}} = -\frac{1}{r^3} \vec{r} \times \vec{J}$ $\dot{\varphi} \hat{e}_\varphi = \dot{\varphi} \hat{e}_\varphi - \dot{\varphi} \hat{e}_\varphi = 0$



$$e\omega v = \vec{e} \cdot \hat{e}_r = \frac{1}{GM} (\dot{\vec{r}} \times \vec{d}) \cdot \hat{e}_r - 1$$

$$\dot{\vec{r}} \times \vec{d} = (\dot{r}\hat{e}_r + r\dot{\hat{e}}_r) \times (r^2\dot{\hat{e}}_z) = r^2\dot{r}\dot{\hat{e}}_r \times \hat{e}_z + r^3\dot{\hat{e}}_r^2 \times \hat{e}_z$$

$$= r^2\dot{r}\dot{\hat{e}}_r + r^3\dot{\hat{e}}_r^2 \hat{e}_r$$

$$e\omega v = \frac{1}{GM} r^3\dot{v}^2 - 1 =$$

$$\dot{v} = \frac{d}{r^2}$$

$$= \frac{1}{GM} r^3 \frac{d^2}{H^2} - 1$$

$$1 + e\omega v = \frac{d^2}{GM}$$

$$r = \frac{d^2/GM}{1 + e\omega v}$$

$$\frac{m^4 \cdot L}{m^3 s^2 \cdot \frac{1}{K} \cdot \frac{1}{K}} = m \quad e > 0$$

$$d^2/GM = p > 0$$

$$\Rightarrow \left| \frac{p}{1 + e\omega v} \right|$$

$$\hat{e}_r \cdot \vec{e} = e\omega v$$

$$\omega v = \frac{\vec{e} \cdot \hat{e}_r}{e}$$

$$\vec{F} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$$

$$\hat{e}_r, \hat{e}_v$$

$$\dot{\theta} - \dot{v} = 0$$

$$\hat{e}_\theta = \hat{e}_v$$

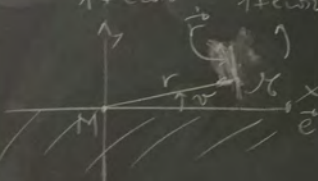
$$d = r^2\dot{\theta} = r^2\dot{v}$$

$$\hat{e}_\theta = (-\sin\theta, \cos\theta)$$

$$\hat{e}_v = (-\sin v, \cos v)$$

S3 - Lezioni del 12 e 14 Ottobre 2016
(Mancano le foto alla lavagna della lezione del 12 Ottobre)

- Classificazione delle orbite e terza legge di Keplero
- Relazione tra semiasse maggiore e integrale dell'energia

$$r = \frac{j^2/GM}{1+e \cos \theta} = \frac{P}{1+e \cos \theta} = \frac{a(1-e^2)}{1+e \cos \theta}$$


$$E = \frac{1}{2} \dot{r}^2 - \frac{GM}{r} = -\frac{GM}{2a} \Rightarrow \dot{r}^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right)$$

$$E(r, \dot{r}) = \frac{1}{2} \dot{r}^2 + \frac{1}{2} \frac{j^2}{r^2} - \frac{GM}{r} =$$

$$E(r, \dot{r}) = \frac{1}{2} \dot{r}^2 + \left(\frac{j^2}{2r^2} - \frac{GM}{r} \right) = \frac{1}{2} \dot{r}^2 + U(r)$$

$$j^2 = GM a (1-e^2)$$

$$e^2 = 1 + \frac{2 E j^2}{G^2 M^2}$$

$$E = -\frac{GM}{2a}$$

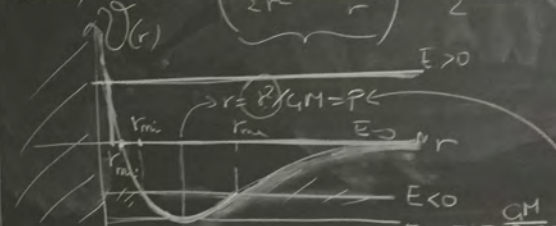
$$M = m_1 + m_2$$

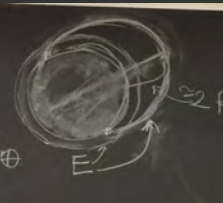
$$jL = \frac{m_1 m_2}{M}$$

$$0 \leq e < 1 \quad a > 0 \text{ finito} \quad E < 0 \quad \text{ellisse/circolo}$$

$$e = 1 \quad a \rightarrow \infty \quad E = 0 \quad \text{parabola}$$

$$e > 1 \quad a < 0 \text{ finito} \quad E > 0 \quad \text{iperbole}$$



$$E = -\frac{GM}{2a}$$


$$a_{\text{sputnik}} \approx R_{\oplus}$$

$$l = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} \dot{\theta}$$
 Integrale primo del moto

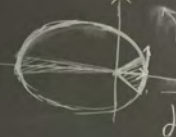
$$\frac{dl}{dt} = \frac{1}{2} r^2 \frac{d\dot{\theta}}{dt}$$
 velocità angolare

$$0 \leq e < 1 \quad \text{moto periodico}$$

$$| \dot{\theta} P = \pi a b | \Rightarrow P = \frac{\pi a^2 \sqrt{1-e^2}}{\frac{1}{2} l} = \frac{2\pi a^2 \sqrt{1-e^2}}{GM a^{1/2} \sqrt{1-e^2}} = \frac{2\pi a^{3/2}}{(GM)^{1/2}}$$

$$n = \frac{2\pi}{P} \quad \text{moto medio}$$

$$n^2 a^3 = GM$$
 3^a legge KEPLERO $G_1 (m_1 + m_2)$


 2^a legge di Keplero

$$\dot{\theta} = \vec{r} \times \dot{\vec{r}} = \text{costante}$$

$$\frac{P^2}{a^3} = \frac{4\pi^2}{GM}$$

- Orbita nello spazio e definizione degli elementi kepleriani
- Definizione di anomalia eccentrica, relazioni con l'anomalia vera, equazione di Keplero e sua soluzione con il metodo di Newton

6 Elementi kepleriani:

$a, e, i, \omega, \Omega, v(t)$

ω argomento del pericentro

a, e

$v(t)$

linea dei nodi
 Ω longitudine del nodo ascendente

$0 \leq \Omega < 2\pi$

$0 \leq i < \pi$

$0 \leq \omega < 2\pi$

b

$r(t) = \frac{a(1-e^2)}{1+e\cos v(t)}$

$v(t) \rightarrow r(t)$

$n^2 a^3 = GM$

$n = \frac{2\pi}{P}$ moto medio

$n(t-t_p) = \ell(t)$

$\ell(t)$ anomalie eccentrica

$\ell(t)$ anomalie media

$r \cos u = r \cos v + ae$

$a \sqrt{1-e^2} \sin u = r \sin v$

$r \cos v = \frac{a(1-e^2)}{1+e\cos v} - \frac{r}{1+e\cos v}$

$ea \cos v = \frac{a(1-e^2)}{1+e\cos v} - \frac{r}{1+e\cos v} + \frac{e}{1+e\cos v}$

$ea \cos u = a - \frac{e^2}{1+e\cos v} - \frac{r}{1+e\cos v}$

$$r = \frac{a(1-e^2)}{1-e\cos\theta} \Rightarrow \frac{r}{a} = \frac{1-e\cos\theta}{1-e^2}$$

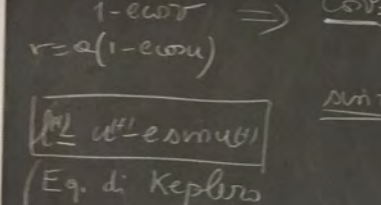
$$r = a(1-e\cos\theta)$$

$$\boxed{u = u_0 - e \sin u_0}$$

(Eq. di Keplero)

$$n(t-t_p) = u(t) - e \sin u(t)$$

$$t=0 \rightarrow n t_p = u(t_p) - e \sin u(t_p)$$

$$\Rightarrow t_p = - \frac{l(0)}{n}$$


$$u - e \sin u - l = 0$$

$$\Rightarrow n(t-t_p) = u - e \sin u$$

[illegible]

$$\begin{cases} \cos u(0) = \frac{1}{e} \left(1 - \frac{r(0)}{a}\right) \\ \sin u(0) = \frac{\dot{r}(0)}{e \dot{u}(0)} = \frac{\dot{r}(0) r(0)}{e n a} \end{cases}$$

$$\downarrow u(0)$$

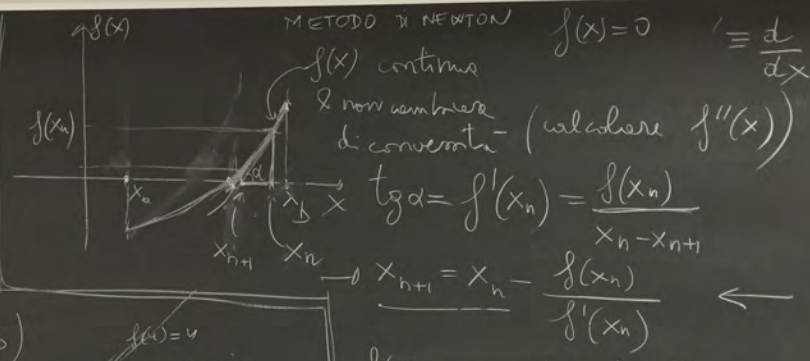
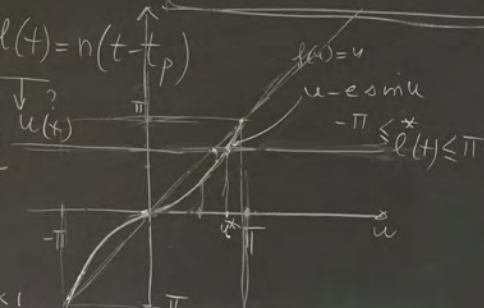
$$l(0) = u(0) - e \sin u(0)$$

$$\downarrow t_p = -\frac{l(0)}{n} \Rightarrow l(t) = n(t - t_p)$$

$$l(t) = u(t) - e \sin u(t)$$

Eq. di Keplero

$$t - \pi \leq l(t) \leq \pi \quad 0 < e < 1$$



$$f(u) = u - e \sin u - l = 0$$

$$u_{n+1} = u_n - \frac{u_n - e \sin u_n - l}{1 - e \cos u_n}$$

S5 - Lezioni del 26 e 28 Ottobre 2016

- Formulario per la soluzione numerica di problema dei due corpi tramite l'equazione di Keplero
- Rudimenti di FORTRAN

Handwritten notes on two blackboards detailing the solution of the two-body problem using Kepler's equation.

Top Blackboard:

- Left side:**
 - Diagram of two bodies M_1 and M_2 with masses m_1, m_2 and total mass $M = m_1 + m_2$.
 - Position vectors $\vec{r}(0), \dot{\vec{r}}(0)$ and $\vec{r}(t), \dot{\vec{r}}(t)$.
 - Angular momentum $\vec{J} = \vec{r} \times \dot{\vec{r}}$ and its magnitude $J = |\vec{J}|$.
 - Unit vectors $\hat{e}_1, \hat{e}_2$ and the eccentricity vector $\vec{e} = \frac{1}{GM} \dot{\vec{r}}(0) \times \vec{J} - \frac{\vec{r}(0)}{r(0)}$.
 - Diagram of the orbit in the $\hat{e}_1, \hat{e}_2$ plane.
- Right side:**
 - Kepler's equation: $l(t) = n(t - t_p)$ and $l(t) = u(t) - e \sin u(t)$.
 - Mean motion $n = \frac{2\pi}{P}$.
 - Energy $E = -\frac{GM}{2a}$ and $n^2 a^3 = GM$.
 - Diagram of the orbit with semi-major axis a and eccentricity e .

Bottom Blackboard:

- Left side:**
 - Kepler's equation: $l(t) = l(0) + n \Delta t$ and $l(t) = u(t) - e \sin u(t)$.
 - Diagram of the orbit with true anomaly u and eccentric anomaly u .
- Right side:**
 - Position and velocity components: $x = r \cos v = (a \cos u - ae)$, $y = r \sin v = a \sqrt{1-e^2} \sin u$.
 - Velocity components: $\dot{x} = -a \dot{u} \sin u$, $\dot{y} = a \sqrt{1-e^2} \dot{u} \cos u$.
 - Diagram of the orbit with true anomaly v and eccentric anomaly u .

FORTRAN (FORmula TRANslator)

Varabili: 6 caratteri (alfanumerici) inizia: lettera

A-H, O-Z reale

I, J, K, L, M, N intero

+, -, *, /

* *

→ A=B

X=X+1

DIMENSION A(N) ¹ continue

FUNCTION

SUBROUTINE

↑ Program KEPLER

DOUBLE PRECISION FUNCTION PRSCAL (a, b)

IMPLICIT DOUBLE PRECISION (A-H, O-Z)

dimension a(3), b(3)

prscal = 0.0d0

do 1 i=1,3
prscal = prscal + a(i)*b(i)

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a} \cdot \vec{b}$$

↑ subroutine procc (a, b, c)

c

a(1)=a(2)*b(3)-a(3)*b(2)

c(2)=a(3)*b(1)-a(1)*b(3)

c(3)=a(1)*b(2)-a(2)*b(1)

return

end

$\vec{a}, \vec{b}$

$\alpha \vec{a} + \beta \vec{b}$

vline (alfa, a, beta, b, c)

$$\frac{1}{\sin(\vec{r} \times \vec{j})} = \frac{1}{r} \vec{r}$$

subroutine ankep(r, rp, gm, eps, ek, en)

dimension r(3), rp(3), ek(2), vz(3), gei(3), e(3)

r, rp, gm, eps: INPUT; ek=(n, e, 1, omega pauli, omega

grande, emmelle media, a), en OUTPUT

data pig, vz/3.14...d0, d.d0, a.d0, i.d0/

call procc(r, rp, gei)

geis = size(gei)

cosi = gei(3)/geis

ek(3) = dcos(cosi)

call procc(rp, gei, e) $\vec{r} \times \vec{j} \rightarrow \vec{e}$

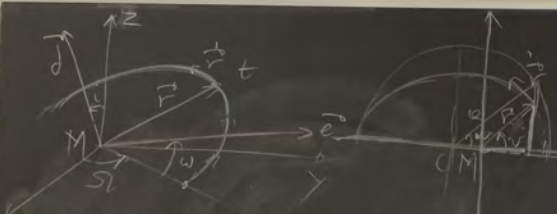
rs = size(r)

call vline(1.d0/gm, e, -1.d0/rs, r, e)

ek(2) = size(e)

- Subroutines, Functions e programma principale per la soluzione numerica del problema dei due corpi tramite soluzione dell'equazione di Keplero.
- Uso delle rotazioni in 2 dimensioni per il passaggio dal piano del moto all'orbita ai vettori posizione e velocità istantanea nel riferimento inerziale in 3D di partenza

[illegible]



$l(t) = l(0) + n \Delta t$
 $\vec{l}^{(1)} = \vec{u}^{(1)} - e \sin u^{(1)} \hat{e}$
 $\vec{e}k(6)$
 $\vec{e}k(2)$

subroutine kepler (ek, gm, eps, t, rp, en)
 - $u = \text{eccar}(\vec{e}k(2), \vec{e}k(6), \text{eps})$
 - $\cos u = \text{dcos}(u)$
 - $\sin u = \text{dsin}(u)$
 - $\text{beta} = \text{dsqrt}(1.0 - \vec{e}k(2) \times \vec{e}k(2))$
 $x(1) = \vec{e}k(2) \times (\cos u - \vec{e}k(2))$
 $x(2) = \vec{e}k(2) \times \text{beta} \times \sin u$
 $x(3) = 0.0$

$rp(3) = v(3)$
 call rota (v, ek(5), rp)
 $v2 = \text{pascal}(rp, rp)$
 $en = v2 / 2.0 - gm / rs$
 return
 end

call rota (x, ek(4), x)
 call rota (x(2), ek(3), x(2))
 $r(3) = x(3)$
 call rota (x, ek(5), t)
 $rs = \text{size}(r)$
 $rma = 1.0 - \vec{e}k(2) \times \cos u$
 $rma = rs / \vec{e}k(2)$
 $up = \vec{e}k(1) / rma$
 $v(1) = -\vec{e}k(2) \times up \times \sin u$
 $v(2) = \vec{e}k(2) \times \text{beta} \times up \times \cos u$
 $v(3) = 0.0$
 call rota (v, ek(4), v)
 call rota (v(2), ek(3), v(2))

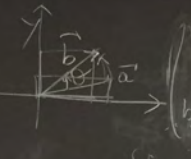
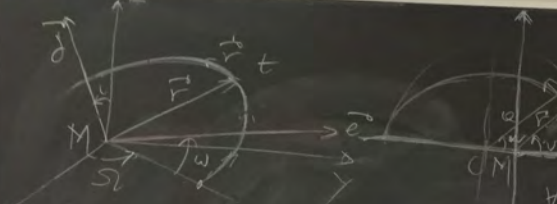
subroutine rota (a, teta, b)
 note: il vettore a di + teta (cos l) -
 note: gli assi di - teta, vettore b
 input - -
 dimension a(2), b(2), w(2)
 $\cos t = \text{dcos}(teta)$
 $\sin t = \text{dsin}(teta)$
 $w(1) = a(1) \times \cos t - a(2) \times \sin t$
 $w(2) = a(1) \times \sin t + a(2) \times \cos t$
 $b(1) = w(1)$
 $b(2) = w(2)$
 return
 end

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{cases} x = a \cos u - ae \\ y = a \sqrt{1-e^2} \sin u \end{cases} \quad \begin{cases} b_1 = a_1 \cos t - a_2 \sin t \\ b_2 = a_1 \sin t + a_2 \cos t \end{cases}$$

$$\begin{cases} \dot{x} = -a \dot{u} \sin u \\ \dot{y} = a \sqrt{1-e^2} \dot{u} \cos u \end{cases} \quad \dot{u} = n \frac{a}{r}$$

$r(1) = a(1 - e \cos u)$

program kepler
 implicit double precision a-h, o-z
 dimension r(3), rp(3), ek(2)
 write (*,*) 'tphi'
 read (*,*) tphi
 eps = 1.d-15
 gm = 1.d6

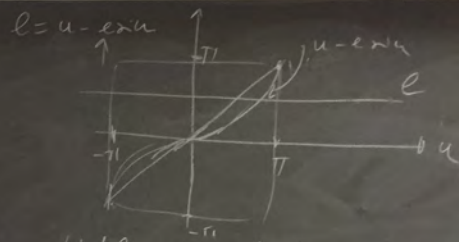
$l(t) = l(0) + n(t - t_0)$
 Δt

write (*,*) 'vettore posizione relativa a t0'
 read (*,*) r
 write (*,*) 'vettore velocità relativa a t0'
 read (*,*) rp
 call curkep (r, rp, gm, eps, ek, en)
 $\text{comp} = \vec{e}k(6)$
 do 1 i = 1, 10
 write (*,*) 't1'
 read (*,*) t1
 $\text{deltat} = t1 - tphi$
 $\vec{e}k(1) = \text{comp} + \vec{e}k(1) \times \text{deltat}$
 call kepler (ek, gm, eps, r, rp, en)
 write (*,*) 't1, r, rp, en'
 continue


```

double princi function ecca(e, am, eps)
implent
pij = 4.0 * atan(1.0)
am = princi(am)
u = pij * segno(am)
do 1 i=1, 20
d = (u - e * dn(u) - am) / (1.0 - dn(u))
u = u - d
if (dabs(d).lt. eps) then
    ecca = u
    return
endif
1 continue return end

```



```

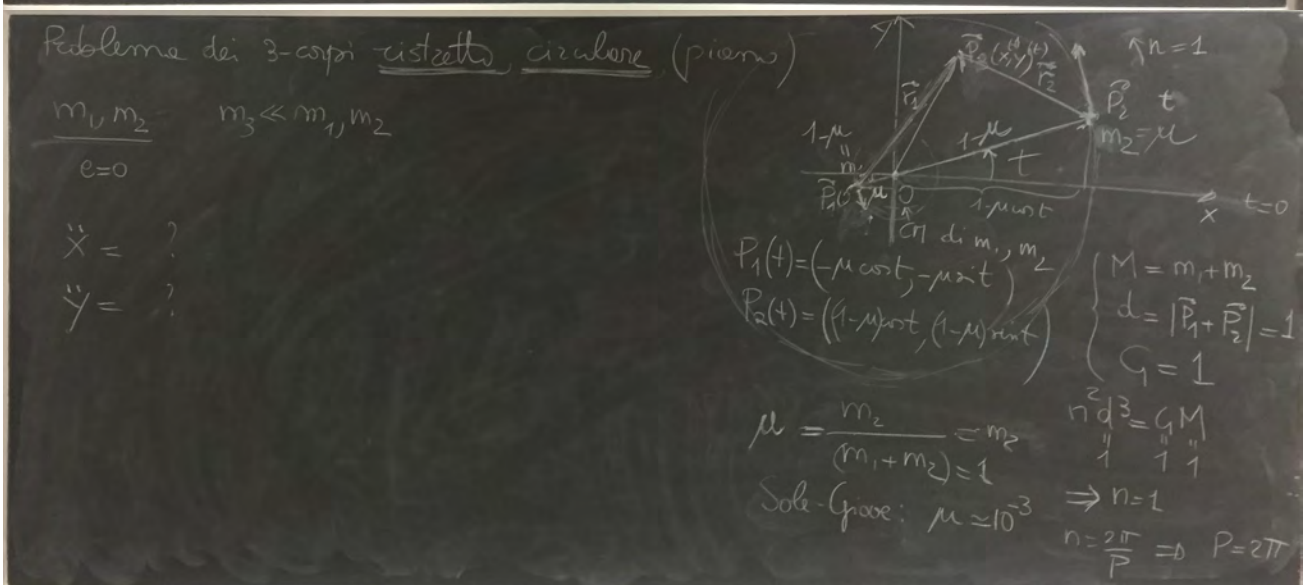
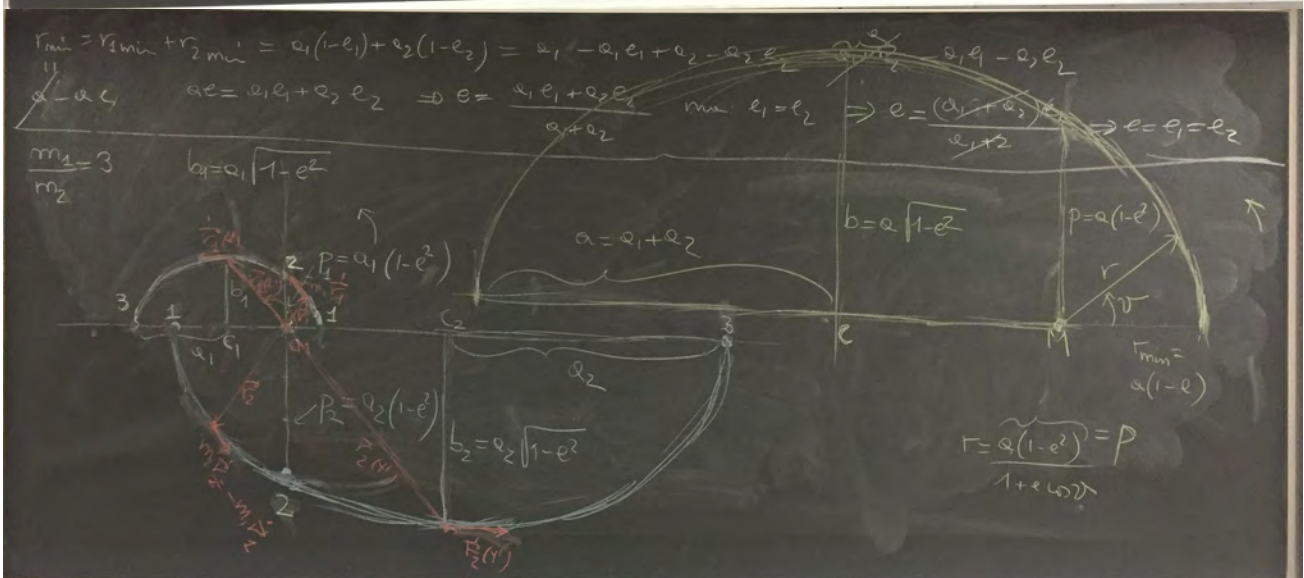
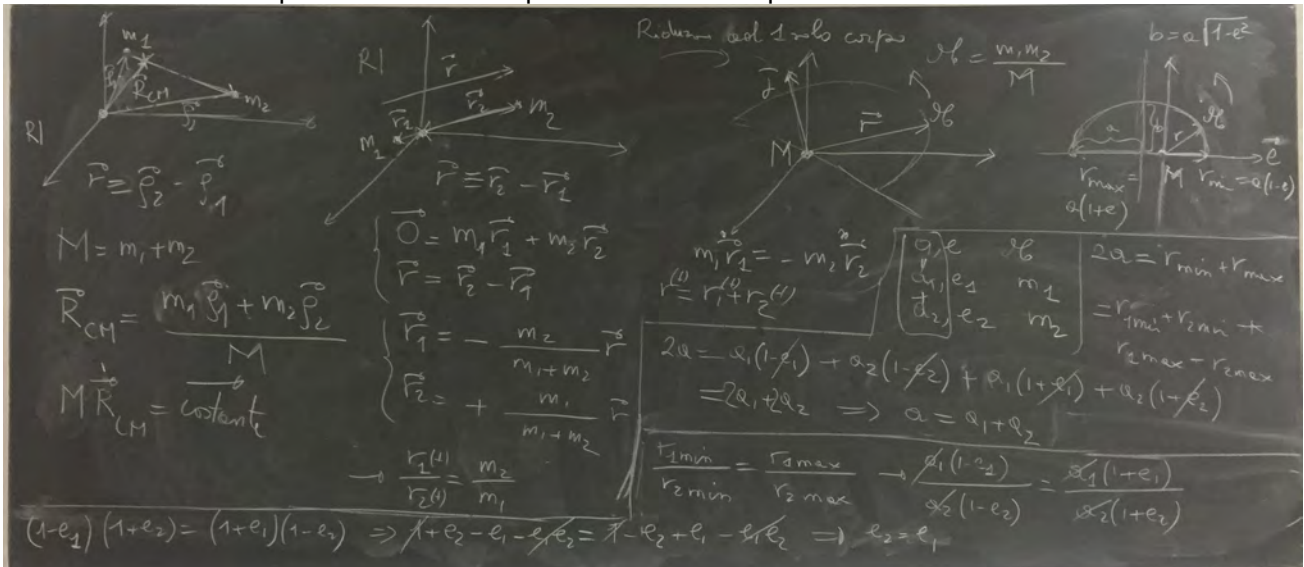
double princi function princi(teta)
pij = ...
dpij = 2.0 * pij
npij = dn(dn(teta)) / dpij
teta = teta - (npij * dpij * segno(teta))
if (teta.lt. -pij) teta = teta + dpij
if (teta.gt. pij) teta = teta - dpij
prigi = teta
return prigi

```

S7 - Lezioni del 9 e 11 Novembre 2016

(Mancano le foto alla lavagna dell'11 novembre)

- Passaggio dall'orbita soluzione del problema ridotto ad un solo corpo ad un solo corpo alle orbite dei singoli corpi
- Formulazione del problema dei tre corpi ristretto circolare piano



- Problema dei tre corpi circolare piano: equazioni del moto nel riferimento inerziale; equazioni del moto nel riferimento rotante con la binaria; integrale di Jacobi; curve a velocità zero
- Posizione del punto di equilibrio lagrangiano L1 situato tra il primario e il secondario della binaria lungo la linea che li congiunge

Diagram showing a two-body system in a rotating frame. The primary body is at the origin, and the secondary body is at position $\vec{r}_2$. The rotating frame is defined by unit vectors $\hat{x}$ and $\hat{y}$. The position of the secondary body in the rotating frame is $\vec{r}_2 = (x, y)$. The equations of motion in the rotating frame are:

$$\ddot{x} = 2\dot{y} + \ddot{x} - \frac{\mu}{r_2^3}(x-1+\mu) - \frac{1-\mu}{r_1^3}x$$

$$\ddot{y} = -2\dot{x} + \ddot{y} - \frac{\mu}{r_2^3}y - \frac{1-\mu}{r_1^3}y$$

The Jacobi integral is given by:

$$C = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\mu}{r_2} - \frac{1-\mu}{r_1} = \text{costante}$$

The position of the Lagrangian point L1 is found by setting the derivative of the potential to zero:

$$\frac{d}{dx} \left[\frac{\mu}{r_2} + \frac{1-\mu}{r_1} \right] = 0$$

Equations for the Jacobi integral and the potential function:

$$\frac{d}{dx} \left[\frac{\mu}{r_2} \right] = -\frac{\mu}{r_2^3}(x-1+\mu)$$

$$\frac{d}{dy} \left[\frac{\mu}{r_2} \right] = -\frac{\mu}{r_2^3}y$$

$$\frac{d}{dx} \left[\frac{1-\mu}{r_1} \right] = -\frac{1-\mu}{r_1^3}x$$

$$\frac{d}{dy} \left[\frac{1-\mu}{r_1} \right] = -\frac{1-\mu}{r_1^3}y$$

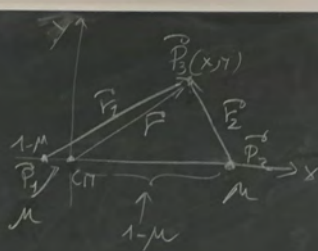
The potential function is:

$$2C(x, y) = \dot{x}^2 + \dot{y}^2 - \frac{\mu}{r_2} - \frac{1-\mu}{r_1} + (\dot{x}y - \dot{y}x)$$

The Jacobi integral is:

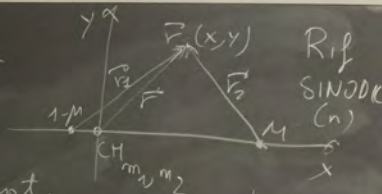
$$C(x, y) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{\mu}{r_2} - \frac{1-\mu}{r_1} + (\dot{x}y - \dot{y}x)$$

X/Y ^{Prima e poi} Riferimento rotante con
la binomia ($n=1$)



$$C = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega(x, y)$$

$$S(x,y) = \frac{1}{2}(x^2+y^2) + \frac{m}{r_2} + \frac{q-m}{r_1}$$



- Calcolo della posizione del punto di equilibrio di L1 usando il concetto di marea. Definizione di sfera di influenza
- Calcolo dell'integrale di Jacobi della curva a velocità zero passante per il punto L1
- Criterio di stabilità di Hill e sua applicazione agli asteroidi tra Marte e Giove
- Definizione di marea e calcolo del potenziale mareale generato da un corpo puntiforme sulla superficie di un corpo esteso in un sistema di due corpi in orbita circolare

Handwritten notes on a chalkboard showing the derivation of the L1 point position and the Jacobi integral for a two-body system.

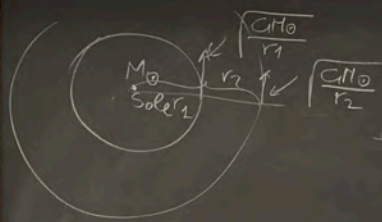
Left side: Diagram of two masses m_1 and m_2 separated by distance D . A point L_1 is located at distance d from m_2 . The distance d is much smaller than D ($d \ll D$). The gravitational force from m_1 at L_1 is $\frac{Gm_1}{(D+d)^2}$. The centrifugal force is $n^2(D+d)$. The equilibrium condition is $\frac{Gm_1}{(D+d)^2} + n^2(D+d) = 0$. The angular velocity n is given by $n^2 D^3 = Gm_1$. The position of L_1 is $d_{L1} \approx \left(\frac{\mu}{3}\right)^{1/3} D$.

Right side: Diagram of the coordinate system with m_1 at $(-1, 0)$ and m_2 at $(\mu, 0)$. The position of L_1 is (x, y) . The Jacobi integral is $J = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}r^2 + \frac{\mu}{r_2} + \frac{1-\mu}{r_1}$. The value of J_{L1} is $J_{L1} \approx 1.52$. The parameters are $G=1$, $m_1+m_2=1$, $D=1$, and $\mu = \frac{m_2}{m_1+m_2} = 10^{-3}$.

Handwritten notes on a chalkboard showing the derivation of the Jacobi integral and the position of the L1 point for a two-body system.

Left side: Diagram of the coordinate system with m_1 at $(-1, 0)$ and m_2 at $(\mu, 0)$. The position of L_1 is (x, y) . The Jacobi integral is $J = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}r^2 + \frac{\mu}{r_2} + \frac{1-\mu}{r_1}$. The value of J_{L1} is $J_{L1} \approx 1.52$. The parameters are $G=1$, $m_1+m_2=1$, $D=1$, and $\mu = \frac{m_2}{m_1+m_2} = 10^{-3}$.

Right side: Diagram of the coordinate system with m_1 at $(-1, 0)$ and m_2 at $(\mu, 0)$. The position of L_1 is (x, y) . The Jacobi integral is $J = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}r^2 + \frac{\mu}{r_2} + \frac{1-\mu}{r_1}$. The value of J_{L1} is $J_{L1} \approx 1.52$. The parameters are $G=1$, $m_1+m_2=1$, $D=1$, and $\mu = \frac{m_2}{m_1+m_2} = 10^{-3}$.



$$J(\alpha) = \frac{1-\mu}{2\alpha} + \alpha^{1/2}(1-\mu)^{1/2} - \alpha\mu + \frac{\mu}{1-\alpha} + O(\mu^2)$$

($\mu \approx 10^{-3}$) $0.3 \leq \alpha \leq 0.93$

α	$J(\alpha)$
0.30	2.214
0.35	2.020
0.40	1.882
0.45	1.782
0.50	1.707
0.55	1.651
0.60	1.609
0.65	1.576
0.70	1.552
0.75	1.535
0.80	1.523
0.81	1.520

THULE

$$\alpha = 0.85 \rightarrow J(\alpha) = 1.515 < J_{L1}$$

CRITERIO DI STABILITA' DI HILL
APPLICATO AGLI ASTEROIDI TRA MARTE
E GIOVE

$$J_{L1} = 1.520$$

DISTINZIONE TRA ORBITE E CURVE
A VELOCITA' ZERO

m, r $\vec{R}$ corpo generatore di marea

$\Phi_G(P) - \Phi_G(O) = \phi_{\text{tide}}$

$\epsilon = \frac{r^2}{R^2} - \frac{2r}{R} \cos \lambda$

$(1+\epsilon)^{-1/2} \approx 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 \approx$
 $-\frac{1}{2}(1+\epsilon)^{-3/2} \approx 1 - \frac{1}{2}\frac{r^2}{R^2} + \frac{r}{R} \cos \lambda$
 $+\frac{3}{4}(1+\epsilon)^{5/2} \approx 1 + \frac{3}{2}\frac{r^2}{R^2} \cos^2 \lambda$

$\phi_G(P) = \frac{GM}{|\vec{R}-\vec{r}|} \approx \frac{GM}{(R^2 + r^2 - 2Rr \cos \lambda)^{1/2}} =$
 $= \frac{GM}{R(1 + \frac{r^2}{R^2} - \frac{2r}{R} \cos \lambda)^{1/2}} \approx$
 $\approx \frac{GM}{R} \left[1 + \frac{r}{R} \cos \lambda + \frac{1}{2} \frac{r^2}{R^2} (3 \cos^2 \lambda - 1) - O\left(\frac{r^3}{R^3}\right) \right]$

m, r $\vec{R}$ corpo generatore di marea

$\Phi_G(P) - \Phi_G(O) = \phi_{\text{tide}}$

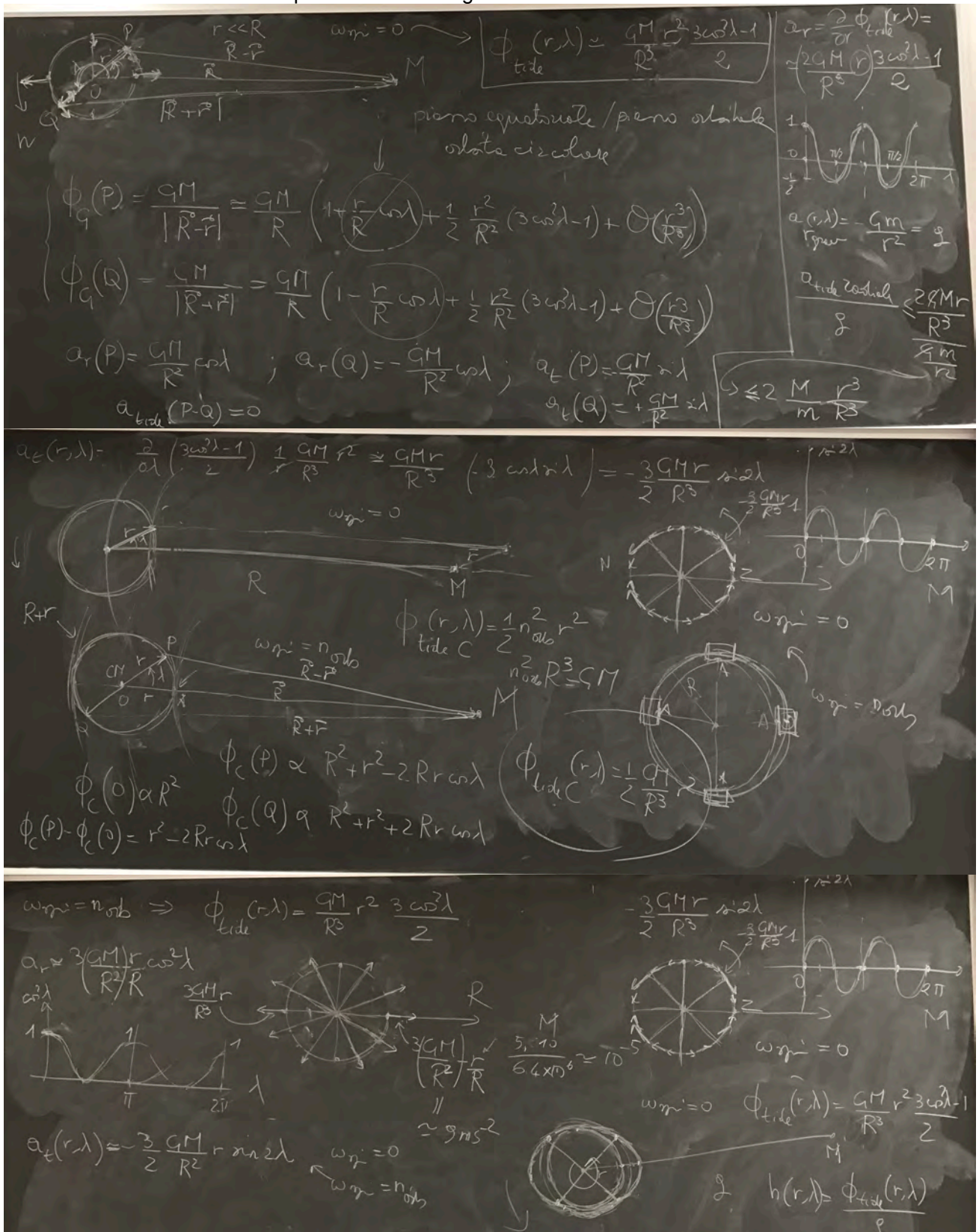
$\epsilon = \frac{r^2}{R^2} - \frac{2r}{R} \cos \lambda$

$(1+\epsilon)^{-1/2} \approx 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 \approx$
 $-\frac{1}{2}(1+\epsilon)^{-3/2} \approx 1 - \frac{1}{2}\frac{r^2}{R^2} + \frac{r}{R} \cos \lambda$
 $+\frac{3}{4}(1+\epsilon)^{5/2} \approx 1 + \frac{3}{2}\frac{r^2}{R^2} \cos^2 \lambda$

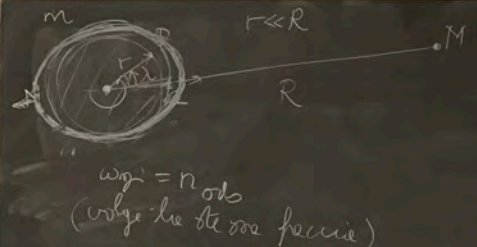
$\phi_G(P) = \frac{GM}{|\vec{R}-\vec{r}|} \approx \frac{GM}{(R^2 + r^2 - 2Rr \cos \lambda)^{1/2}} =$
 $= \frac{GM}{R(1 + \frac{r^2}{R^2} - \frac{2r}{R} \cos \lambda)^{1/2}} \approx$
 $\approx \frac{GM}{R} \left[1 + \frac{r}{R} \cos \lambda + \frac{1}{2} \frac{r^2}{R^2} (3 \cos^2 \lambda - 1) - O\left(\frac{r^3}{R^3}\right) \right]$

La lezione del 2 dicembre non si è tenuta per impegni della docente (tempo recuperato allungando al durata delle lezioni successive)

- Calcolo delle accelerazioni di marea, componenti radiale e componente trasversale
- Effetti maresimali derivanti dal potenziale centrifugo

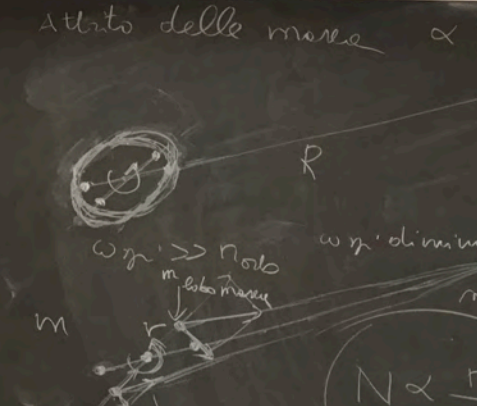


- Altezza della marea e superficie di equilibrio in presenza di marea
- Attrito della marea, diversi tipi di evoluzione orbitale derivanti dall'attrito di marea. Caso di sistemi co-rototanti (satelliti che rivolgono sempre la stessa faccia al proprio pianeta, come la Luna alla Terra) ed evoluzione mareale successiva alla co-rotazione (la Terra rallenta la sua rotazione propria e la distanza Terra-Luna aumenta)
- Il caso di Mercurio intorno al Sole
- Mercurio e Venere non possono avere satelliti perché l'attrito di marea del sole su questi pianeti li rallenta e causa l'evoluzione orbitale distruttiva dei loro eventuali satelliti

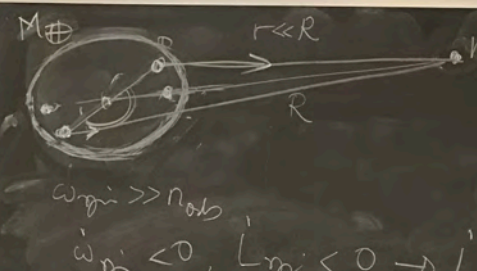


m $r \ll R$ M n_{orb} $\phi_{tide}(r, \lambda) = \left(\frac{GM}{R}\right) \frac{r^2}{R^2} \frac{3\cos^2\lambda - 1}{2}$ angolo fisso
 $\omega_{rot} = n_{orb}$ (volge la stessa faccia)
 $g(r, \lambda) = -\frac{GM}{r^2} = g$
 $\phi_{tide}(r, \lambda) = g h(r, \lambda)$
 $h(r, \lambda) = \frac{1}{g} \frac{GM}{R} \frac{r^2}{R^2} \frac{3\cos^2\lambda - 1}{2} = \frac{M}{m} \frac{1}{R^3} \frac{3\cos^2\lambda - 1}{2}$ angolo fisso
 $h(r, \lambda) = \frac{1}{g} \frac{GM}{R} \frac{r^2}{R^2} \frac{3\cos^2\lambda}{2} = \frac{M}{m} \frac{1}{R^3} \frac{3\cos^2\lambda}{2}$
 $a_r(r, \lambda) = \frac{2}{r} \phi_{tide}(r, \lambda)$
 $a_t(r, \lambda) = \frac{1}{r} \frac{\partial \phi_{tide}(r, \lambda)}{\partial \lambda} \rightarrow a_t(r, \lambda) = -\frac{3}{2} \frac{GM}{R^3} r \sin 2\lambda$

Attrito delle maree $\propto 1/R^6$



M n_{orb} R m $\omega_{rot} > n_{orb}$ ω_{rot} diminuisce e ω_{orb} aumenta
 $N \propto \frac{r^5}{R^6}$
 $L_{tot} = L_{rot} + L_{orb} = \text{costante}$ $L_{rot} < 0 \Rightarrow L_{orb} > 0$
 $\dot{L}_{orb} = -\dot{L}_{rot}$



$M \oplus$ $r \ll R$ M n_{orb} $\omega_{rot} > n_{orb}$
 $\dot{\omega}_{rot} < 0$, $\dot{L}_{rot} < 0 \Rightarrow \dot{L}_{orb} > 0$
 la distanza terra-luna aumenta
 $h_0 \propto \frac{2}{g} \frac{GM_0 R_0^2}{d_{\oplus}^3}$
 $h_r \propto \frac{2}{g} \frac{GM_r R_0^2}{d_{\oplus}^3}$
 $\frac{h_r}{h_0} = \frac{M_r}{M_0} \frac{d_{\oplus}^3}{d_{\oplus}^3} \approx 2.5$

$\omega_{gr} = 0$ (corretto fino nello spazio)

$$\phi(r, \lambda) = \frac{GM}{R^3} r^2 \frac{3\cos^2\lambda - 1}{2}$$

$$h(r, \lambda) = \frac{1}{g} \phi(r, \lambda) = \frac{r^2}{4M} \cdot \frac{GM}{R^3} \frac{3\cos^2\lambda - 1}{2} = \left(\frac{M}{m} \frac{r^3}{R^3} \right) r \frac{3\cos^2\lambda - 1}{2}$$

$$a_r(r, \lambda) = 2 \frac{GM}{R^3} r \frac{3\cos^2\lambda - 1}{2}$$

$$a_t(r, \lambda) = -\frac{3}{2} \frac{GM}{R^3} r \sin 2\lambda$$

$$h_{max} = \frac{1}{g} \frac{GM}{R^3} \frac{R^2}{2} \approx 36 \text{ cm}$$

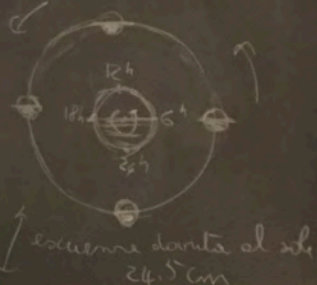
$$h_{min} = 18 \text{ cm}$$

escurs. 54 cm

$$\frac{\phi_{tide, gr}}{\phi_{tide, 0}} = \frac{M_r}{M_0} \frac{d^3 \phi_0}{d^3 \phi} \approx 2.2$$

$$h_{max, 0} = \frac{1}{g} \frac{GM_0 R_0^2}{2} \approx 16.3 \text{ cm}$$

$$h_{min, 0} = \frac{1}{2} h_{max, 0} = 8.2 \text{ cm}$$



SISTEMA ISOLATO



$$N \propto \frac{r^5}{R^6}$$

$$\omega_{gr} > \omega_{orb}$$

$$\dot{\omega}_{gr} < 0, \dot{L}_{gr} < 0 \Rightarrow \dot{L}_{orb} > 0 \Rightarrow \text{aumenta } R$$

$$P_{orb} \approx 88 \text{ d}$$

$$P_{gr} = \frac{2}{3} P_{orb}$$



$$\omega_{gr} < \omega_{orb}$$

$$\dot{\omega}_{gr} > 0, \dot{L}_{gr} > 0 \Rightarrow \dot{L}_{orb} < 0 \Rightarrow \text{si avvicinano}$$

- Derivazione delle equazioni di Eulero per il moto del corpo rigido intorno al proprio centro di massa
- Applicazione delle equazioni di Eulero alla Terra assunta rigida, a forma di sferoide oblato (ellissoide di rivoluzione), isolata nello spazio e rotante attorno ad un asse che passa per il suo centro di massa ma non coincide esattamente con il suo asse di simmetria (calcolo del periodo di precessione libera, o precessione di Eulero)
- Precessione della Terra (rigida, a forma di sferoide oblato, rotante) a causa della Luna e del Sole

CORPO RIGIDO

Diagramma: Un corpo rigido con centro di massa (CM) in un sistema di coordinate (x, y, z) . Si mostra il vettore posizione $\vec{r}_i$ di una particella i rispetto al CM, e il vettore velocità $\vec{v}_i = \vec{\omega} \times \vec{r}_i$. Si indica anche l'angolo di Eulero θ e l'angolo di precessione ϕ .

Equazioni di Eulero:

$$\left(\frac{d\vec{G}}{dt} \right)_{RI} = \vec{\omega} \times \vec{G}$$

$$\left(\frac{d\vec{r}_i}{dt} \right)_{RI} = \vec{\omega} \times \vec{r}_i$$

$$\left(\frac{d\vec{G}}{dt} \right)_{RI} = \vec{\omega} \times \vec{G} + \left(\frac{d\vec{G}}{dt} \right)_{BF}$$

Nota: $\left(\frac{d\vec{G}}{dt} \right)_{BF}$ in generale non nulla.

Definizioni:

- $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}$
- $\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i$
- $\vec{L} = \sum m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum m_i [\vec{\omega} r_i^2 - \vec{r}_i (\vec{\omega} \cdot \vec{r}_i)]$
- $\vec{L} = \sum m_i r_i^2 \vec{\omega} - \sum m_i \vec{r}_i (\vec{\omega} \cdot \vec{r}_i)$
- $L_x = \omega_x \left[\sum m_i (r_i^2 - x_i^2) \right] - \omega_y \left[\sum m_i x_i y_i \right] - \omega_z \left[\sum m_i x_i z_i \right]$
- $L_y = -\omega_x \left[\sum m_i x_i y_i \right] + \omega_y \left[\sum m_i (r_i^2 - y_i^2) \right] - \omega_z \left[\sum m_i y_i z_i \right]$
- $L_z = -\omega_x \left[\sum m_i x_i z_i \right] - \omega_y \left[\sum m_i y_i z_i \right] + \omega_z \left[\sum m_i (r_i^2 - z_i^2) \right]$

Forma matriciale:

$$\vec{L} = \mathbb{I} \vec{\omega}$$

$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Nota: I_1, I_2, I_3 sono i momenti d'inerzia principali.

Diagramma: Un corpo rigido con centro di massa (CM) in un sistema di coordinate (x, y, z) . Si mostra il vettore posizione $\vec{r}_i$ di una particella i rispetto al CM, e il vettore velocità $\vec{v}_i = \vec{\omega} \times \vec{r}_i$. Si indica anche l'angolo di Eulero θ e l'angolo di precessione ϕ .

$\left(\frac{d\vec{L}}{dt}\right)_{RI} = \vec{N}^{(e)}$
 $\vec{L} = \mathbb{I}\vec{\omega} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$
 $\vec{\omega} \times \vec{L} = \begin{pmatrix} \omega_y I_2 - \omega_z I_1 \\ \omega_z I_3 - \omega_x I_2 \\ \omega_x I_1 - \omega_y I_3 \end{pmatrix} = \begin{pmatrix} (I_2 - I_3)\omega_y \omega_z \\ (I_3 - I_1)\omega_z \omega_x \\ (I_1 - I_2)\omega_x \omega_y \end{pmatrix}$

$\left(\frac{d\vec{L}}{dt}\right)_{BF} + \vec{\omega} \times \vec{L} = \left[\frac{d}{dt}(\mathbb{I}\vec{\omega})\right]_{BF} + \vec{\omega} \times \vec{L} = \mathbb{I}\left(\frac{d\vec{\omega}}{dt}\right)_{BF} + \vec{\omega} \times \vec{L} = \vec{N}^{(e)}$

$\frac{d\mathbb{I}}{dt} = 0$

$\begin{cases} I_1 \dot{\omega}_x - \omega_y \omega_z (I_2 - I_3) = N_x^{(e)} \\ I_2 \dot{\omega}_y - \omega_z \omega_x (I_3 - I_1) = N_y^{(e)} \\ I_3 \dot{\omega}_z - \omega_x \omega_y (I_1 - I_2) = N_z^{(e)} \end{cases}$

$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$
 $\omega_x, \omega_y \ll \omega_z$

$\begin{cases} I_1 \dot{\omega}_x = \omega_y \omega_z (I_1 - I_3) \\ I_2 \dot{\omega}_y = -\omega_z \omega_x (I_3 - I_1) \\ I_3 \dot{\omega}_z = 0 \end{cases}$

$I_3 > I_1$

$\rightarrow$ durata del giorno "costante"

$\omega_z = \text{costante}$
 $\omega_x(t), \omega_y(t)$

$\begin{cases} \dot{\omega}_x = \left[\frac{(I_1 - I_3)}{I_1} \omega_z \right] \omega_y \\ \dot{\omega}_y = - \left[\frac{(I_1 - I_3)}{I_1} \omega_z \right] \omega_x \end{cases}$

$\ddot{\omega}_x = \Omega \dot{\omega}_y = -\Omega^2 \omega_x$
 $\ddot{\omega}_y = -\Omega \dot{\omega}_x = -\Omega^2 \omega_y$

$\begin{cases} \ddot{\omega}_x + \Omega^2 \omega_x = 0 \\ \ddot{\omega}_y + \Omega^2 \omega_y = 0 \end{cases}$

$\Omega = \frac{I_1 - I_3}{I_1} \omega_z \approx 300 \text{ anni}$

$\left| \frac{I_1 - I_3}{I_1} \right| \ll 1$

$\approx 300 \text{ anni}$ (ipotesi Terra = corpo rigido sferico oblat)

$T = \frac{1}{2} \sum_i m_i \vec{r}_i \cdot \dot{\vec{r}}_i = \frac{1}{2} \sum_i m_i \vec{r}_i \cdot (\vec{\omega} \times \vec{r}_i)$

$T = \frac{1}{2} \mathbb{I} \vec{\omega}^2$

$\vec{\omega} = \frac{d\vec{\theta}}{dt}$

Corpo rigido

$3N$ forze N particelle

3 CM

$\begin{cases} +2 \\ +1 \end{cases}$ 6 gradi di libertà

$\begin{cases} +2 \\ +1 \end{cases} \leftarrow$ angoli di Eulero

$\vec{F} = \frac{d\vec{p}}{dt}$ Rif. inerziale

$\vec{p} = (m)\vec{v}$

$\vec{N} = \frac{d\vec{L}}{dt}$ Rif. In

$\vec{L}_i = m_i \vec{r}_i \times \vec{v}_i$

$\vec{L} = \sum_{i=1}^N m_i \vec{r}_i \times \vec{v}_i$

$\vec{v}_i = \left(\frac{d\vec{r}_i}{dt}\right)_{RI} = \left(\frac{d\vec{r}_i}{dt}\right)_{BF} + \vec{\omega} \times \vec{r}_i$

$\vec{v}_i$ rif. inerziale

$\left(\frac{d\vec{r}_i}{dt}\right)_{BF}$ Rif. inerziale

$\vec{\omega} \times \vec{r}_i$

$\left(\frac{d\vec{L}}{dt}\right)_{RI} = \left(\frac{d\vec{L}}{dt}\right)_{BF} + \vec{\omega} \times \vec{L}$

$$\vec{L} = \begin{pmatrix} \sum m_i (y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i x_i y_i & \sum m_i (x_i^2 + z_i^2) & -\sum m_i y_i z_i \\ -\sum m_i x_i z_i & -\sum m_i y_i z_i & \sum m_i (x_i^2 + y_i^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$$

$$\vec{L} = \mathbb{I} \vec{\omega}$$

$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} I_1 \omega_x \\ I_2 \omega_y \\ I_3 \omega_z \end{pmatrix}$$

tenere d'incidenza $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$
 aspetti agli assi principali di inerzia del corpo
 equ. di Eulero

$$\vec{N} = \left(\frac{d\vec{L}}{dt} \right)_{RI} = \left(\frac{d}{dt} (\mathbb{I} \vec{\omega}) \right)_{BF} + \vec{\omega} \times \vec{L} = \mathbb{I} \dot{\vec{\omega}} + \vec{\omega} \times (\mathbb{I} \vec{\omega})$$

$$\begin{cases} N_x = I_1 \dot{\omega}_x - \omega_y \omega_z (I_2 - I_3) \\ N_y = I_2 \dot{\omega}_y - \omega_z \omega_x (I_3 - I_1) \\ N_z = I_3 \dot{\omega}_z - \omega_x \omega_y (I_1 - I_2) \end{cases}$$

$$(\vec{\omega} \times \vec{L})_x = \omega_y L_z - \omega_z L_y = \omega_y I_3 \omega_z - \omega_z I_2 \omega_y = (I_3 - I_2) \omega_y \omega_z$$

$\omega = \frac{2\pi}{86164} \text{ rad s}^{-1}$

$$d\vec{F} = \frac{GM dm}{|\vec{R} - \vec{r}|^2} \frac{(\vec{R} - \vec{r})}{|\vec{R} - \vec{r}|}$$

$$|\vec{R} - \vec{r}|^2 = (R^2 + r^2 - 2Rr \cos \theta)^{3/2}$$

$$|\vec{R} - \vec{r}|^{-3} = (R^2 + r^2 - 2Rr \cos \theta)^{-3/2}$$

$$|\vec{R} - \vec{r}|^{-3} \approx R^{-3} \left(1 + 3 \frac{\vec{R} \cdot \vec{r}}{R^2} \right)$$

$$(1 + \epsilon)^{-3/2} \approx 1 - \frac{3}{2} \epsilon$$

$$d\vec{R} = \vec{r} \times d\vec{F} \approx \frac{GM dm}{R^3} \left(1 + 3 \frac{\vec{R} \cdot \vec{r}}{R^2} \right) \vec{r} \times \vec{R}$$

$$\vec{R} = \int d\vec{R} = \frac{GM}{R^3} \int dm \left(1 + 3 \frac{\vec{R} \cdot \vec{r}}{R^2} \right) \vec{r} \times \vec{R}$$

$$\vec{R} \approx \frac{3GM}{R^5} \int dm (\vec{R} \cdot \vec{r}) (\vec{r} \times \vec{R})$$

$$K_x = \frac{3GM}{R^5} \int (Xx + Yy + Zz) (yZ - zY) dm = \frac{3GM}{R^5} YZ \int (y^2 - z^2) dm = \frac{3GM}{R^5} YZ (I_3 - I_1)$$

$$K_y = -\frac{3GM}{R^5} XZ (I_3 - I_1)$$

$$K_z = \frac{3GM}{R^5} \int dm (Xx + Yy + Zz) (xY - yX) = 0$$

$$I_1 = \int dm (y^2 + z^2)$$

$$I_2 = \int dm (z^2 + x^2)$$

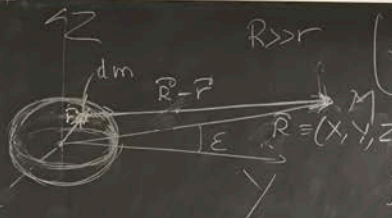
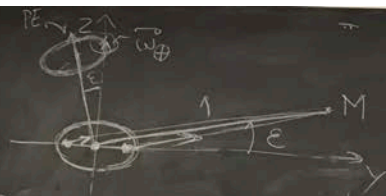
$$I_3 = \int dm (x^2 + y^2)$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \epsilon - x \epsilon \\ 0 & x \epsilon \sin \epsilon \end{pmatrix} \begin{pmatrix} R \\ R \sin \epsilon \\ 0 \end{pmatrix} = \begin{pmatrix} R \cos \epsilon \\ R \sin \epsilon \\ 0 \end{pmatrix}$$

$$\vec{R} \cdot \vec{R} = 0$$

$$K_x X + K_y Y + K_z Z = 0$$

$$\begin{cases} K_x = K_y \\ K_y = K_z \sin \epsilon \\ K_z = -K_y \cos \epsilon \end{cases}$$



$$\Omega = \frac{3GM}{2R^3} \left[\frac{(I_3 - I_1)}{I_3} \right] \frac{1}{\omega_0} \frac{\omega_0}{\omega_0} \frac{ms^{-2}}{ms^{-1}} = s^{-1}$$

$$\Omega_0 + \Omega_1 \quad T_{pre} = \frac{2\pi}{\Omega_0 + \Omega_1} \approx 26 \text{ days}$$

$$K_x = \frac{3GM}{R^3} (I_3 - I_1) \omega_0 \epsilon \sin \epsilon \sin \lambda_0$$

$$K_y = -\frac{3GM}{R^3} (I_3 - I_1) \omega_0 \epsilon \cos \epsilon \sin \lambda_0 \cos \lambda_0$$

$$K_z = \frac{3GM}{R^3} (I_3 - I_1) \omega_0 \epsilon \sin \epsilon \cos \lambda_0$$

$$\vec{R} = (X, Y, Z) = \begin{pmatrix} R \sin \lambda_0 \cos \epsilon \\ R \sin \lambda_0 \sin \epsilon \\ R \cos \lambda_0 \end{pmatrix}$$

$$\frac{M}{d_{or}} \cdot \frac{d^3 \theta_0}{dt^3} \approx 2.2$$

$$K_x = \frac{3GM}{2R^3} (I_3 - I_1) \omega_0 \epsilon \sin \epsilon$$

$$\left(\frac{d\vec{L}}{dt} \right) = \vec{K}$$

$$K_x = \Omega_x L_x - \Omega_y L_y = -\Omega L \sin \epsilon = -\Omega I_3 \omega_0 \epsilon$$

$$K_y = K_z = 0 \quad L = I_3 \omega_0$$

$$\left(\frac{d\vec{L}}{dt} \right)_{BF} + \vec{L} \times \vec{\Omega}$$