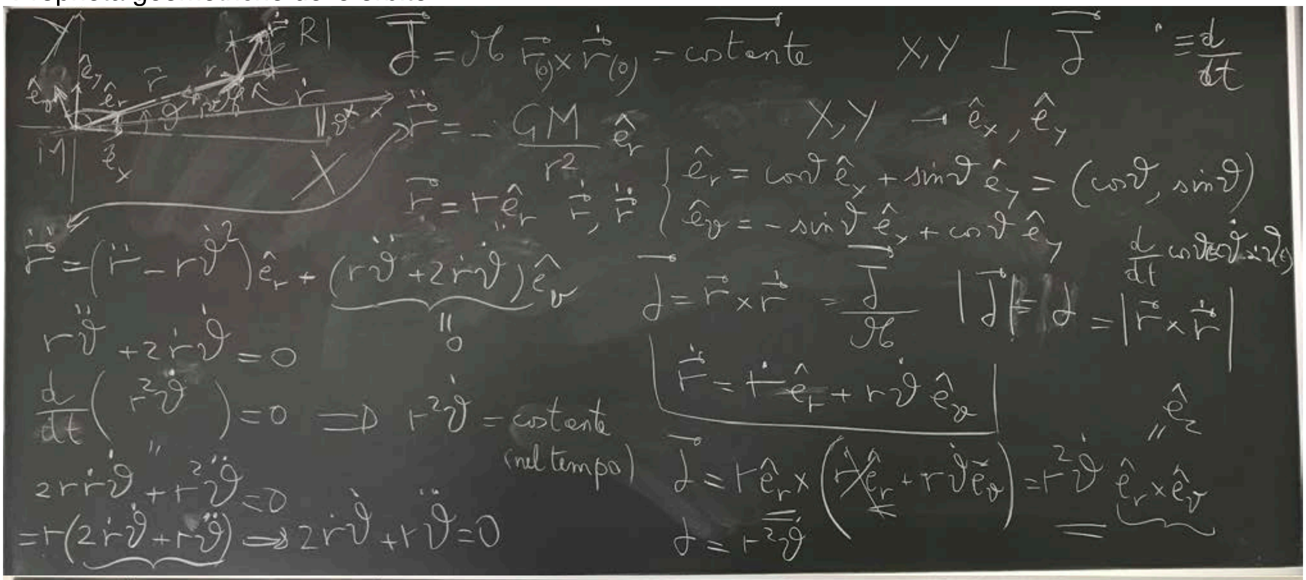
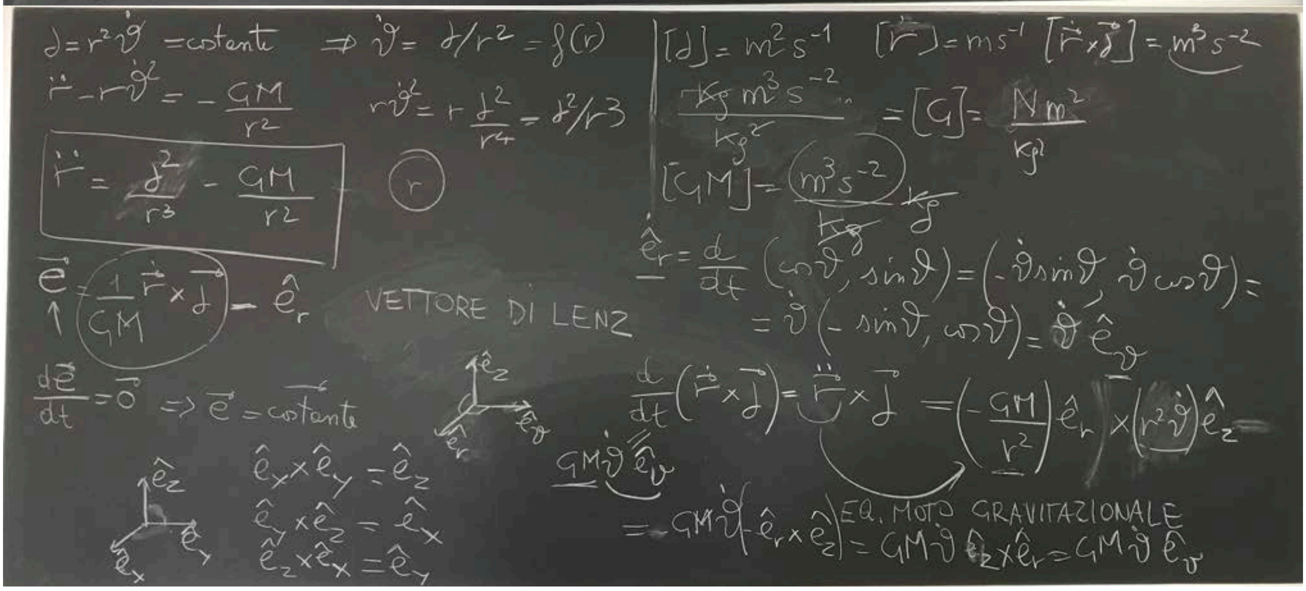
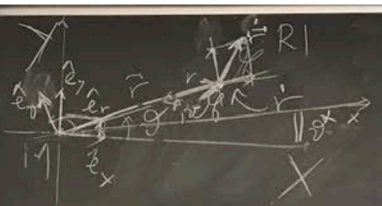
S2: Elementi di Meccanica Celeste, anno accademico 2017-2018, lezioni della prima settimana

2a settimana 2 e 5 ottobre 2017
2 ottobre 16-18
-Equazioni del moto in coordinate polari piane
-Riduzione ad una sola equazione del moto nella variabile scalare r>0 (distanza relativa tra i due corpi ad ogni istante)
-Integrale primo del vettore di Lenz
-Equazione dell'orbita in coordinate polari: le orbite sono coniche
5 ottobre 2017 16-18
-Le orbite sono coniche
-Classificazione delle coniche in base al modulo del vettore di Lenz (eccentricità)
-Proprietà geometriche delle orbite







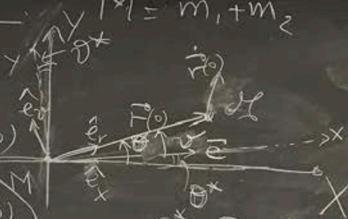
$$\begin{cases} m_1, m_2 \\ \vec r(0),\ \dot{\vec r}(0) \end{cases} \qquad \mu = \frac{m_1 m_2}{M} \qquad M = m_1 + m_2$$

$$X, Y \perp \vec J = \vec r(0) \times \dot{\vec r}(0)$$

$$\boxed{\vec e = \frac{1}{GM}\dot{\vec r}(0) \times \vec J - \frac{\vec r(0)}{r(0)}}$$

$\vec J, \vec e$ costanti

$$\vec J = r^2 \dot\theta \hat e_z = r^2 \dot v \hat e_z$$



$$v = \theta - \theta^* \qquad \dot v = \dot\theta$$

$$\cos\vartheta^* = \frac{\vec e}{e}\cdot \hat e_x \quad \text{costante}$$

v anomalia vera

$$J = r^2 \dot v \Rightarrow \dot v = J/r^2$$

$$\hat e_r \cdot \vec e = 1\, e \cos v$$

$$\vec e \cdot \hat e_r = \frac{1}{GM}\underbrace{(\dot{\vec r} \times \vec J)\cdot \hat e_r}\, - \underbrace{\hat e_r \cdot \hat e_r}_{1}$$

$$\hat e_r \cdot (\dot{\vec r} \times \vec J) = \vec J \cdot (\hat e_r \times \dot{\vec r}) = \vec J \cdot (\hat e_r \times \hat e_v)\, r\dot v = r\dot v (r^2 \dot v)\hat e_z \cdot \hat e_z = r^3 \dot v^2$$

$$\vec J = r^2 \dot v \hat e_z$$

$$\boxed{\vec a \cdot (\vec b \times \vec c) = \vec c \cdot (\vec a \times \vec b)}$$

$$\boxed{\vec e \cdot \hat e_r = e\cos v}$$

$$\boxed{\vec J = r^2 \dot v \hat e_z} \qquad \dot{\vec r} = \dot r \hat e_r + r\dot v \hat e_v$$

$$J = r^2 \dot v \Rightarrow \dot v = J/r^2 \qquad \dot\theta = \dot v$$

$$\left(\frac{1}{GM}\dot{\vec r} \times \vec J - \hat e_r\right)\cdot \hat e_r =$$

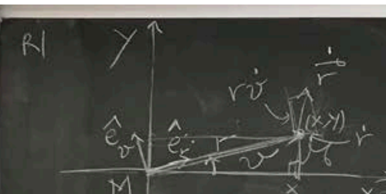
$$= \frac{1}{GM}\dot{\vec r} \times \vec J \cdot \hat e_r - 1 = \frac{1}{GM}\hat e_r \cdot (\dot{\vec r} \times \vec J) - 1 = \frac{1}{GM}\vec J \cdot (\hat e_r \times \dot{\vec r}) - 1 =$$

$$= \frac{1}{GM}(r^2 \dot v)\hat e_z \cdot (r\dot v \underbrace{\hat e_r \times \hat e_v}_{\hat e_z}) - 1 = \frac{1}{GM} r^3 \dot v^2 \cdot 1 - 1 = \frac{r^3 \dot v^2}{GM} - 1 = \frac{r^3 \frac{J^2}{r^4}}{GM} - 1 =$$

$$= \frac{J^2}{GM}\frac{1}{r} - 1$$

$$\frac{J^2/GM}{r} - 1 = e\cos v \;\Rightarrow\; \frac{J^2/GM}{r} = 1 + e\cos v \;\Rightarrow\; \boxed{r(t) = \frac{J^2/GM}{1 + e\cos v(t)}}$$

orbita soluzione del problema dei 2 corpi (ridotto ad 1 solo corpo)



x, y

r, v

$$\mu = \frac{m_1 m_2}{M} \qquad M = m_1 + m_2$$

$$\boxed{\ddot{\vec r} = -\frac{GM}{r^3}\vec r} \quad \times \text{ tutte le forze centrali}$$

$$\vec J = \vec r \times \dot{\vec r} \qquad \dot{\vec J} = 0 \qquad \dot{\vec e} = 0 \;\to\; \propto 1/r^2 \text{ gravitazione}$$

$$\vec e = \frac{1}{GM}(\dot{\vec r} \times \vec J) - \frac{\vec r}{r} \qquad \hat e_r$$

$$\vec r = r\hat e_r$$

$$\dot{\vec r} = \dot r \hat e_r + r\dot v \hat e_v$$

$$J = r^2 \dot v \Rightarrow \dot v = J/r^2$$

$$\boxed{\ddot r = \frac{J^2}{r^3} - \frac{GM}{r^2}}$$

$$\vec e \cdot \hat e_r = e\cos v \;\Rightarrow\; \boxed{r(t) = \frac{J^2/GM}{1 + e\cos v(t)}} \qquad J^2/GM \equiv p$$

$$ms^{-2} = \frac{[GM]}{m^2} \qquad \frac{(m^2 s^{-1})^2}{m^3 s^{-2}} = m$$

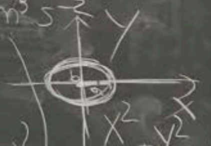
$$x = r\cos v \qquad r = \sqrt{x^2 + y^2}$$

$$r + er\cos v = J^2/GM$$

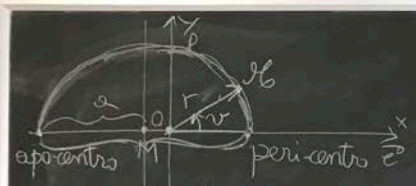
$$r = \frac{J^2}{GM} - er\cos v = p - ex$$

$$(x^2 + y^2)^{1/2} = p - ex \;\Rightarrow\; x^2 + y^2 = (p - ex)^2$$

x asse di simmetria



$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$



$r(v=0) = \frac{p}{1+e}$ $\quad r(v=\frac{\pi}{2}) = p$ $\quad r(v=\pi) = \frac{p}{1-e}$

semilato retto della conica

$\vec e$ indica la direzione del pericentro dell'orbita

e = eccentricità

$$2a = r(v=0) + r(v=\pi) = \frac{p}{1+e} + \frac{p}{1-e} = p\left(\frac{1-e+1+e}{1-e^2}\right) = \frac{2p}{1-e^2} \Rightarrow p = a(1-e^2)$$

$a = p/(1-e^2)$ semiasse maggiore della conica

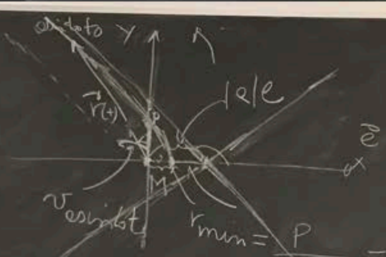
$e=0 \Rightarrow a = p = J^2/GM = r(t) \ \forall t$

$$\begin{cases} 0<e<1 & a \text{ positivo e finito (ellisse)} \\ e=1 & a \to \infty \text{ parabola} \\ e>1 & a<0 \text{ finito iperbole} \end{cases}$$

3) $e>1$ iperbole

$a = \frac{p}{1-e^2}$ $\quad p>0 \Rightarrow a<0$ $\quad |a| = \frac{p}{e^2-1}$

$1 + e\cos v = \frac{p}{r}$



$\boxed{r = \frac{p}{1+e\cos v}}$ $\quad p = \frac{J^2}{GM} = a(1-e^2)$

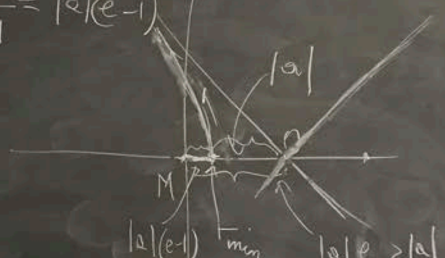
$$\boxed{J^2 = GMa(1-e^2)}$$

$r_{min} = \frac{p}{e+1} = |a|(e-1)$

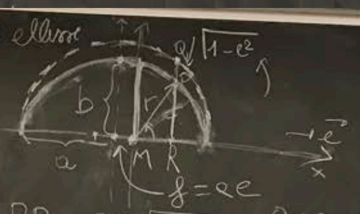
$r_{min} = a(1-e) = |a|(e-1)$

$p = |a|(e^2-1)$

$1 + e\cos v_{asintoto} = 0 \Rightarrow \cos v_{asint} = -\frac{1}{e}$ $\quad e>1$



1) ellisse



$PR = QR\sqrt{1-e^2}$ $\quad 0 \le e < 1$

$b = a\sqrt{1-e^2}$

$r_{min} = r(v=0) = \frac{a(1-e^2)}{1+e} = a(1-e)$

$r_{max} = r(v=\pi) = \frac{a(1-e^2)}{1-e} = a(1+e)$

$r_{max} - r_m = a(1+e) - a(1-e) = a + ae - a + ae = 2ae$

$$r = \frac{J^2/GM = p = a(1-e^2)}{1+e\cos v}$$

$$\boxed{J^2 = GMa(1-e^2)}$$

$x^2 + y^2 = (p-ex)^2$

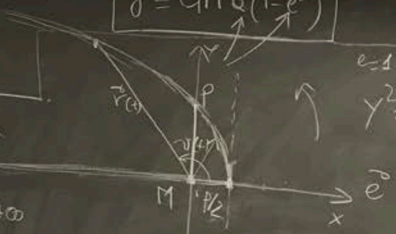
$e=1$ $\quad \cancel{x^2} + y^2 = p^2 - 2x + \cancel{x^2}$

$y^2 = p^2 - 2x$

2) parabola

$e=1$

$a = \frac{p}{1-e^2} \to \infty$



$x' = x - p/2$ $\quad x'' = -x'$ $\quad r_{min} = \frac{p}{2}$