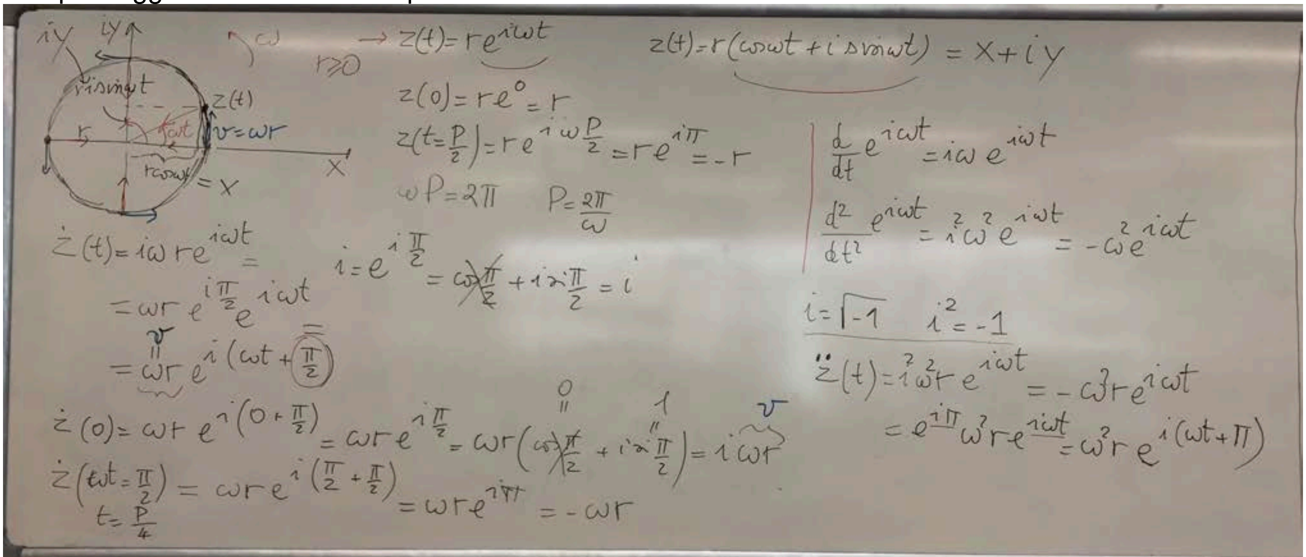
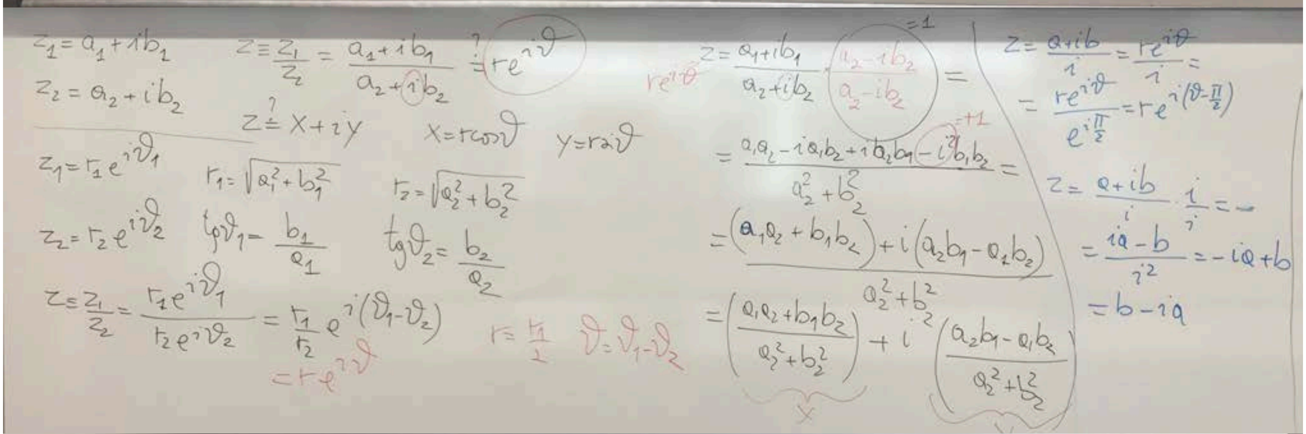
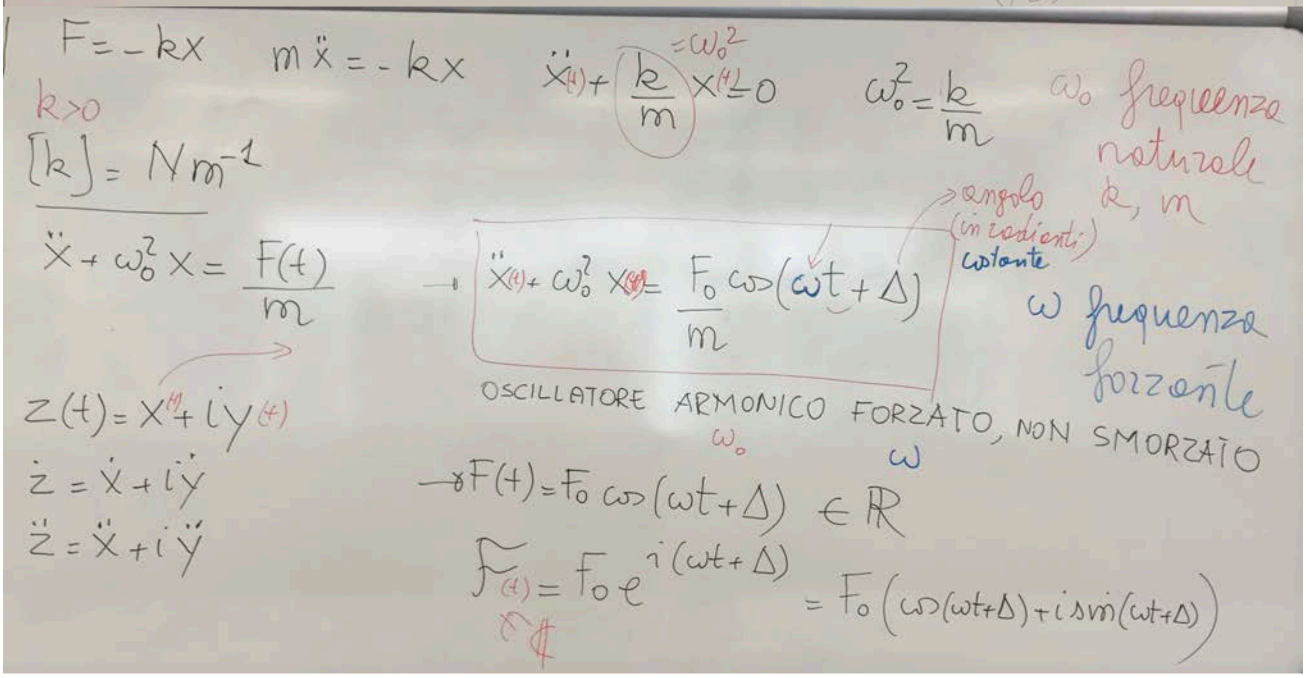
S6 - Sesta settimana, lezioni del 21 e 23 marzo 2017

- Esercizi sui numeri complessi con riferimento alla loro rappresentazione geometrica. Moto circolare uniforme nel piano complesso e moti oscillatori lungo l'asse reale e quello immaginario
- Oscillatore armonico forzato (non smorzato) in una dimensione: equazione del moto e soluzione con passaggio alla variabile complessa
- Oscillatore armonico smorzato (non forzato) in una dimensione: equazione del moto e soluzione con passaggio alla variabile complessa







$\boxed{\ddot{z}(t) + \omega_0^2 z(t) = \frac{\tilde{F}}{m}} \xrightarrow{\mathbb{R}} \ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t+\Delta) = \frac{F}{m}$

$\overset{\parallel}{\frac{F_0}{m}} e^{i(\omega t+\Delta)}$

? radianti

$\omega_0^2 = \frac{k}{m}$

→ possibile soluzione: $z(t) = \overset{?}{a}\, e^{i(\omega t+\delta)}$

m

$\dot{z} = i\omega a e^{i(\omega t+\delta)}$ m

$\ddot{z} = i^2\omega^2 a e^{i(\omega t+\delta)} = -\omega^2 a e^{i(\omega t+\delta)}$

$-\omega^2 a e^{i(\omega t+\delta)} + \omega_0^2 a e^{i(\omega t+\delta)} = \frac{F_0}{m} e^{i(\omega t+\Delta)}$

$-\omega^2 a e^{i\omega t} e^{i\delta} + \omega_0^2 a e^{i\omega t} e^{i\delta} = \frac{F_0}{m} e^{i\omega t} e^{i\Delta}$

$\times e^{-i\omega t}$ $= e^0 = 1$, $\times e^{-i\omega t}$

$-\omega^2 a e^{i\delta} + \omega_0^2 a e^{i\delta} = \frac{F_0}{m} e^{i\Delta}$

$e^{i\delta}(\omega_0^2-\omega^2)a = \frac{F_0}{m} e^{i\Delta}$

$\boxed{a e^{i\delta}} = \frac{F_0/m}{\omega_0^2-\omega^2} e^{i\Delta}$

$-1 = e^{i\pi}$

$\delta = \Delta + \pi$

$\omega = \omega_0 \Rightarrow a \to +\infty$ ← m

risonanza — assenza di dissipazione

$a = \frac{F_0/m}{|\omega_0^2-\omega^2|}$

Se $\omega^2 < \omega_0^2 \Rightarrow \delta = \Delta$

Se $\omega^2 > \omega_0^2$: $a e^{i\delta} = -\left(\frac{F_0/m}{\omega^2-\omega_0^2}\right) e^{i\Delta} = e^{i\pi}\left(\frac{F_0/m}{\omega^2-\omega_0^2}\right) e^{i\Delta} = \frac{F_0/m}{\omega^2-\omega_0^2} e^{i(\Delta+\pi)}$

http://eotvos.dm.unipi.it/homenobili.html

25 Maggio - compitino

OSCILLATORE ARMONICO FORZATO 1D

$m\ddot{x} = -kx + F(t)$

$F(t) = F_0\cos(\omega t+\Delta)$

$m\ddot{x} + kx = 0 \longrightarrow \omega_0^2 = k/m$

$m\ddot{x} + kx = F_0\cos(\omega t+\Delta)$

$\rightarrow \ddot{x}(t) + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t+\Delta)$ eq. REALE

$\rightarrow \ddot{z}(t) + \omega_0^2 z(t) = \frac{F_0}{m} e^{i(\omega t+\Delta)}$

EQ COMPLESSA

$$\begin{cases} z = x+iy \\ F(t) = F_0 e^{i(\omega t+\Delta)} = F_0\left(\cos(\omega t+\Delta) + i\sin(\omega t+\Delta)\right) \end{cases}$$

possibile soluzione

$z(t) = \overset{?}{a}\, e^{i(\omega t+\overset{?}{\delta})}$

$a, \delta \in \mathbb{R}$

$\dot{z}(t) = a i \omega e^{i(\omega t+\delta)}$

$\ddot{z}(t) = a i^2\omega^2 e^{i(\omega t+\delta)} = -\omega^2 a e^{i(\omega t+\delta)}$

$(-\omega^2 + \omega_0^2)\, a e^{i(\omega t+\delta)} = \frac{F_0}{m} e^{i(\omega t+\Delta)}$

$e^{i\omega t} e^{i\delta} \times e^{-i\omega t}$; $e^{i\omega t} e^{i\Delta} \times e^{-i\omega t}$

$a e^{i\delta} = \frac{F_0/m}{\omega_0^2-\omega^2} e^{i\Delta}$; $\in \mathbb{R}$; $\in \mathbb{R}$

$[a] = m$; $F_0/m \rightarrow m\,s^{-2}$

$a = \frac{F_0/m}{|\omega_0^2-\omega^2|}$ date

SE $\omega < \omega_0$

$\omega_0^2 - \omega^2 > 0$

$a = \frac{F_0/m}{\omega_0^2-\omega^2} > 0$; $F = F_0\cos(\omega t+\Delta)$

$\delta = \Delta$

$$\boxed{z(t) = \frac{F_0/m}{\omega_0^2-\omega^2} e^{i(\omega t+\Delta)}}$$

ampiezza acc. forzante

$x(t) = \frac{F_0/m}{\omega_0^2-\omega^2}\cos(\omega t+\Delta)$

SE

$\omega > \omega_0$ $\quad a = \frac{F_0/m}{\omega^2-\omega_0^2} > 0$ $\qquad -1 = e^{i\pi}$

$z(t) = -\frac{F_0/m}{\omega^2-\omega_0^2} e^{i(\omega t+\Delta)} =$ ($-= e^{i\pi}$, >0)

$= \frac{F_0/m}{\omega^2-\omega_0^2} e^{i(\omega t+\Delta+\pi)}$

$x(t) =$

SE

$\omega = \omega_0$ $\quad a \rightarrow +\infty$

OSCILLATORE ARMONICO SMORZATO (non forzato) 1D

$F_{diss} = -\gamma\dot{x}$ $\quad \boxed{PO}\ m\ddot{x} = -kx - \gamma\dot{x}$

$(\omega_0^2 = k/m)$ $\qquad [\gamma] = \frac{N}{m\,s^{-1}} = \frac{kg\, m\, s^{-2}}{m\, s^{-1}} = kg\, s^{-1}$

$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = 0$ Eq. omogenea

$\frac{\gamma}{m} = \lambda > 0$; $\frac{k}{m} = \omega_0^2$

$[\omega_0] = rad\, s^{-1}$

$\omega_0 = 2\pi\nu_0 = \frac{2\pi}{P_0}$ $\quad \nu_0 = \frac{1}{P_0}$ $\quad \left[\frac{\gamma}{m}\right] = s^{-1}$

$\ddot{x} + \lambda\dot{x} + \omega_0^2 x = 0$ $\qquad z(t) = x(t) + iy$

$\rightarrow \ddot{z} + \lambda\dot{z} + \omega_0^2 z = 0$

eq. moto in forma complessa

$z(t) = \alpha e^{\beta t}$ possibile soluzione

$\beta \in \mathbb{C}$ parte reale + parte immaginaria

$\alpha = a e^{i\delta}$ $\quad a \in \mathbb{R}, \delta \in \mathbb{R}$ $\quad \alpha \in \mathbb{C}$

$\dot{z} = \alpha\beta e^{\beta t}$

$\ddot{z} = \alpha\beta^2 e^{\beta t}$

$\alpha\beta^2 e^{\beta t} + \lambda\alpha\beta e^{\beta t} + \omega_0^2\alpha e^{\beta t} = 0$

$\alpha e^{\beta t}(\beta^2 + \lambda\beta + \omega_0^2) = 0 \Rightarrow \beta^2 + \lambda\beta + \omega_0^2 = 0$

$$\boxed{\beta_\pm = -\frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} - \omega_0^2}}$$

SE $\frac{\lambda}{2}=\omega_0$ $\quad \beta = -\frac{\lambda}{2}$ reale, <0 $\quad z(t)=\alpha e^{-\frac{\lambda}{2}t} = a e^{i\delta} e^{-\frac{\lambda}{2}t} \Rightarrow \boxed{x(t)=a\cos\delta\, e^{-\frac{\lambda}{2}t}}$

Smorzamento critico

Decadimento esponenziale con costante di tempo τ

$\tau=\frac{2}{\lambda}$

$x(t)=a\cos\delta\, e^{-t/\tau}$

$t=\tau \rightarrow x(\tau)=a\cos\delta\,\frac{1}{e}$

$t=2\tau \quad x(2\tau)=a\cos\delta\,\frac{1}{e^2}$

$\vdots$

$t=n\tau \quad x(n\tau)=a\cos\delta\,\frac{1}{e^n}$

SE $\frac{\lambda}{2}>\omega_0$ $\quad \beta_{\pm}=-\frac{\lambda}{2}\pm\sqrt{\frac{\lambda^2}{4}-\omega_0^2} < 0$

OSCILLATORE SOVRASMORZATO

$z_{\pm}(t)=a e^{i\delta} e^{\beta_{\pm}t}$ esponenziale reale decrescente ($\beta_{\pm}$ negativo)

SE $\frac{\lambda}{2}<\omega_0$ $\quad \beta_{\pm}=-\frac{\lambda}{2}\pm i\sqrt{\omega_0^2-\frac{\lambda^2}{4}}$ >0 $=\omega<\omega_0$

OSCILLATORE SOTTOSMORZATO

$z_{\pm}(t)=a\, e^{i\delta}\, e^{-\frac{\lambda}{2}t}\, e^{\pm i\omega t} = a\, e^{-\frac{\lambda}{2}t}\, e^{i(\pm\omega t+\delta)}$

$x_{\pm}(t)=a\, e^{-t/\tau}\cos(\pm\omega t+\delta)$