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Stabilization of Weakly Coupled Rotors: a General Derivation of the Required Forces

Donato Bramanti, Anna M Nobili and Giuseppe Catastini
Gruppo di Meccanica Spaziale, Dipartimento di Matematica
Università di Pisa, Pisa, Italia
nobili@dm.unipi.it

Summary

A system of two weakly coupled rotors with natural frequency of relative oscillation $\omega_\circ$ much smaller than the spin frequency ω_{spin} and non zero dissipation (due to friction inside its rotating parts, referred to as *rotating damping*) is known to develop a forward whirling motion of increasing amplitude. The force necessary to stabilize the system (by preventing the whirling motion from growing) is derived on the basis of general physical principles with no assumption on the nature and the amount of dissipation in the system. It is found that, even in the presence of very high viscous friction the stabilizing force is smaller than the elastic spring force which couples the system. In all other cases (structural damping only, or structural damping plus small-to-medium viscous damping) the frequency of the destabilizing whirling motion is essentially the natural frequency of the system and the stabilizing force is only $1/Q_{spin}$ of the spring force, Q_{spin} being the quality factor of the springs which accounts for all dissipation (structural plus viscous), measured at the spin frequency. In the GG case, where the spin frequency is $5\,Hz$ at which very high Q_{spin} can be achieved (as we have already found in laboratory tests), the required stabilizing force is therefore smaller than the spring force by far. Application of the right amount of force by means of active electrostatic dampers which spin together with the rotors does in no way change the amount of stabilizing force to be provided; this has also been checked in a thorough numerical simulation of the GG system carried out by Alenia Spazio, including drag disturbance and implementation errors, and with a very conservative assumption on the amount of dissipation in the system. We conclude that: *i)* the stabilizing forces required in the GG experiment (provided by electrostatic actuators spinning with the system) are a factor about 10^6 smaller than the value claimed by Y. Jafry and M. Weinberger in their *Appendix* to the ESTEC Technical Assessment of GG; *ii)* even in the presence of a very large amount of viscous damping the stabilizing forces would be smaller than the spring forces and never dominate the system. We notice that Y. Jafry and M. Weinberger have assumed the damping coefficient of the system (whose physical dimensions are *mass/time*) to depend on the reference frame, which is clearly incorrect in Galilean mechanics. Moreover, they appear to have misunderstood *non rotating friction* with *friction in the bearings*, claiming that the system is stabilized by friction in the bearings (or by an active simulation of it), while the most efficient way of stabilizing the system is well known to be the non rotating friction (or an active simulation of it). From the fact that the active damping forces are much smaller than the forces of the springs it follows that the essentially passive nature of the GG space experiment at a rotation rate of $5\,Hz$ is confirmed. Extremely weak mechanical coupling and good balancing of the test masses (a common mode rejection of $5 \cdot 10^{-6}$ has already been achieved with a ground prototype) are very advantageous for testing the Equivalence Principle, as it is the signal modulation at $5\,Hz$.

1. A Two Dimensional Harmonic Oscillator

Consider a 2 dimensional harmonic oscillator made of two equal point masses, each of mass m, coupled by a spring of stiffness $k/2$ (which corresponds to the stiffness k for an equivalent spring connecting each mass to the centre of mass of the system). The general solution $\vec{r}(t) = (x(t), y(t))$ (for each mass) is an elliptical orbit with the centre (not the focus) in the common centre of mass of the bodies:

$$\begin{cases} x(t) = A\cos(\omega_\circ t - \phi_\circ) \\ \\ y(t) = B\sin(\omega_\circ t - \phi_\circ) \end{cases} \tag{1}$$

where the x, y coordinate axes are chosen to coincide with the symmetry axes of the ellipse, $\omega_\circ = \sqrt{(k/2)/(m/2)} = \sqrt{k/m}$ is the natural frequency of the oscillation, $m/2$ is the reduced mass of the system, $\phi_\circ$ is the phase, A and B are the amplitudes along the symmetry axes. This general solution can be decomposed in various ways into the sum of two simple harmonic motions. For instance, it can be written as the sum of two circular oscillations, one forward and the other backward (for each mass). They have the same frequency and phase but in general different ammplitudes:

$$\vec{r}_f = \rho_f e^{i(\omega_\circ t - \phi_\circ)} \qquad , \qquad \vec{r}_b = \rho_b e^{-i(\omega_\circ t - \phi_\circ)} \tag{2}$$

where ρ_f ρ_b are the amplitudes of the forward and backward circular oscillations respectively:

$$\rho_f = \frac{A+B}{2} \qquad , \qquad \rho_b = \frac{A-B}{2} \tag{3}$$

2. A 2-D Oscillator with Dissipation: Calculation of the Damping Force

Consider the dissipation of the whole system as expressed by the quality factor Q. Q accounts for internal dissipation in the suspensions, for losses due to imperfect clamping and for any other possible losses in the system. The total energy of the system will decrease in time, from its initial value $E(0)$, as follows:

$$E(t) = E(0)e^{-\omega_\circ t/Q} = E(0)e^{-t/\tau} \tag{4}$$

where τ, defined by $\tau \equiv Q/\omega_\circ$, is the damping time of the system due to the dissipation. If this damping time is much longer than the natural period $T_\circ = 2\pi/\omega_\circ$ of the oscillator (i.e. if $\tau \gg T_\circ$), then, in one natural period $T_\circ$, the energy decrease is:

$$\frac{(\Delta E)_{T_\circ}}{E} \simeq -\frac{2\pi}{Q} \tag{5}$$

which is the usual definition for the quality factor Q. The amplitude of the circular oscillation will decrease accordingly:

$$r(t) = r(0)e^{-\omega_\circ t/(2Q)} \tag{6}$$

and the relative variation of the amplitude of oscillation, in one natural period, is:

$$\frac{(\Delta r)_{T_\circ}}{r} \simeq -\frac{\pi}{Q} \tag{7}$$

By monitoring the amplitude of oscillation the Q of the system can be measured, thus measuring its total dissipation. This decrease in the amplitude of the circular oscillation can be interpreted as due to a decrease of the along track velocity, which in turn can be considered as caused by an average damping acceleration a_d, also along track, such that:

$$\frac{1}{2}a_d T_\circ^2 = 2\pi(\Delta r)_{T_\circ} \quad , \quad a_d \simeq -\frac{1}{Q}\omega_\circ^2 r \tag{8}$$

Finally, we can consider a_d as produced by an average damping force F_d on each mass:

$$F_d = ma_d \simeq -\frac{1}{Q}m\omega_\circ^2 r = -\frac{1}{Q}F_c = \frac{1}{Q}kr = -\frac{1}{Q}F_{spring} \tag{9}$$

where $F_c = m\omega_\circ^2 r$ is the centrifugal force, equal and opposite to the elastic force of the spring $F_{spring} = -kr$. The damping force F_d is at about 90° with respect to F_{spring}. We have:

$$|F_d| \simeq \frac{1}{Q}|F_{spring}| \tag{10}$$

Thus, if $Q \gg 1$ the damping force is much smaller than the elastic force of the spring. Note that Q is the experimentally measured value, thus accounting for all losses in the system. Equation (10) also gives the "destabilizing" force, i.e. the force that would be necessary, in absence of damping, to produce an exponential increase, with quality factor $-Q$, of the amplitude of the oscillation. If such "destabilizing" force is applied to the damped oscillator it prevents the exponential decrease of the amplitude of oscillation by pumping into the oscillator exactly the same amount of energy it dissipates; the energy provided at any time is the same as in (4), but with the plus sign in the exponential, so as to produce an undamped oscillator. Also the converse is true: if an external damping force F_d, of magnitude given by (10), is applied to an unstable oscillator having a quality factor $-Q$, it will prevent in the same way the exponential increase of the amplitude of the oscillations, that is the oscillator will be stabilized.

3. A System of Two Weakly Coupled Rotors with Rotating Friction: Whirling Motion and Destabilizing Force

Let now the two bodies of the oscillator be concentric cylinders rotating with angular velocity ω_{spin}. Assume that they are mechanically coupled by a spring with a stiffness k so small that the natural frequency of relative oscillation $\omega_\circ = \sqrt{k/m}$ is much smaller than ω_{spin}. It is known that in this case each body rotates around its own symmetry axis and there is a position of relative equilibrium (fixed in the rotating frame) very close to the spin axis. Assume for the time being perfect centring (i.e. the equilibrium position lies exactly on the spin axis).

If there is friction inside rotating parts of the system (e.g. the springs) this amounts to a non–zero *rotating damping* which has to be taken into account in the equations of motion. Rotating damping has a destabilizing effect because it produces a spin down of the system and a corresponding (forward) whirling motion of the rotating bodies around their common centre of mass, with an exponentially increasing amplitude. The angular frequency of whirl ω_w depends on the kind of rotating damping present in the system. Let Q represent the total dissipation of the system due to friction inside its rotating parts. We can distinguish between structural damping and viscous damping. Structural damping (also known as hysteresis damping) is due to the relative motions of different parts in the material when subject to deformations (the springs); the particles maintain essentially their relative positions and the motions are due to the deformations. Instead, viscous damping occurs between particles sliding the ones with respect to the others.

The frequency of whirl is computed by solving the equations of motion with rotating friction in the system being of structural nature and of viscous nature (see Appendix).

In the case of rotating structural friction we find (using $\omega_\circ/\omega_{spin} \ll 1$) a whirling angular frequency:

$$\omega_w = \frac{\omega_\circ}{\sqrt{2}}\sqrt{1+\sqrt{1+\frac{1}{Q^2}}} \tag{11}$$

yielding, for $Q \gg 1$,

$$\omega_w \simeq \omega_\circ\big(1+\frac{1}{8Q^2}\big) \simeq \omega_\circ \tag{12}$$

That is, the frequency of the destabilizing (forward) whirling motion is essentially the natural frequency of the system. In the case of rotating viscous friction, expressed by a quality factor Q_v of the system, we consider, for a rotor with a given ratio $\omega_\circ/\omega_{spin} \ll 1$, three subcases: of very small, intermediate and very large viscous damping, namely: $Q_v \gg \omega_{spin}/\omega_\circ$; $Q_v \simeq \omega_{spin}/\omega_\circ$; $Q_v \ll \omega_{spin}/\omega_\circ$;. The resulting whirling frequencies are:

$$\omega_w = \omega_\circ \big(1 + \frac{\omega_{spin}^2}{8\omega_\circ^2 Q_v^2}\big) \simeq \omega_\circ \qquad (Q_v \gg \frac{\omega_{spin}}{\omega_\circ}) \tag{13}$$

$$\omega_w \simeq \omega_\circ \sqrt{\frac{1+\sqrt{2}}{2}} \simeq 1.1\omega_\circ \qquad (Q_v \simeq \frac{\omega_{spin}}{\omega_\circ}) \tag{14}$$

$$\omega_w = \omega_\circ \sqrt{\frac{\omega_{spin}}{2\omega_\circ Q_v}} \gg \omega_\circ \qquad (Q_v \ll \frac{\omega_{spin}}{\omega_\circ}) \tag{15}$$

It is therefore apparent that, unless the system has a very large coefficient of rotating damping due to viscous friction (with a viscous quality factor much smaller than the ratio spin–to–natural frequency) the frequency of the whirling (destabilizing), motion is close to the natural frequency of the system (Eqs. (13) and (14)). A small amount of viscous damping does not change the frequency of whirl in any significant way with respect to the case in which only structural damping is present, while in the presence of a very large viscous damping it would be $\omega_w \gg \omega_\circ$. In the GG experiment we can certainly exclude the presence of very large viscous friction (see Appendix), and therefore it is $\omega_w \simeq \omega_\circ$.

We can now compute the quality factor Q_w of the whirling motion, which is defined by the equation:

$$r_w(t) = r_w(0)e^{-\omega_w t/(2Q_w)} \tag{16}$$

where now

$$Q_w < 0 \tag{17}$$

because r_w increases with time as the whirling motion gains angular momentum, necessarily from the spin angular momentum of the rotor. Consider a system of two concentric hollow cylindrical rotors, each of mass m (section across the spin axis), coupled by weak springs and rotating at $\omega_{spin} \gg \omega_\circ$ (Fig. 1). They develop a forward whirling motion of radius r_w around the common centre of mass O and at angular frequency ω_w ($\omega_w \simeq \omega_\circ$ except in the presence of very large viscous friction in which case it is $\omega_\circ \ll \omega_w \leq \omega_{spin}$). For our calculations the elastic properties of the system can be represented by a spring subdivided into 4 springs at 90° from one another, each with longitudinal stiffness $k/4$. This system of springs is equivalent to two springs with stiffness $k/2$ coupling the two masses in both the x and y directions.

The time variation of the spin angular velocity of the system can be computed from the conservation of the total angular momentum, namely the angular momentum of spin:

$$L_{rotor} \simeq 2mR^2\omega_{spin} \tag{18}$$

(R is the linear dimension of the rotor) plus the angular momentum of the whirling motion:

$$L_w = 2mr_w^2\omega_w \tag{19}$$

It must be:

$$\dot{L}_{rotor} + \dot{L}_w = 0 \tag{20}$$

Hence:

$$\dot{\omega}_{spin} = -\frac{2r_w\omega_w}{R^2} \cdot \dot{r}_w \tag{21}$$

From (16) the rate of growth $\dot{r}_w(t)$ is:

$$\dot{r}_w = -\frac{\omega_w}{2Q_w} r_w \tag{22}$$

and the corresponding despin rate of the rotor is:

$$\dot{\omega}_{spin} = \frac{1}{Q_w}\frac{r_w^2}{R^2}\omega_w^2 \qquad (Q_w < 0) \tag{23}$$

which gives the time variation of the spin angular velocity of the rotor in terms of the "negative" dissipation of the whirling motion.

Let us now consider the energy of the system. Since the springs are very weak and their masses are negligible compared to the mass of the rotor (see Fig. 1), they will be obliged to follow the motion of the attachement points which rotate at ω_{spin} around the centre of mass of the respective test mass. The centres of mass of the springs will rotate around O at ω_{spin}. When the springs are going from position 1 to position 3 (see Fig. 1) they will be forced to expand by $4r_w$, and when going from position 3 to 1 to contract by the same amount. In the figure the four position numbers represent the phase of the whirling motion, in $90°$ steps, with position 1 always in the direction O_1O_2. They rotate around O with angular frequency ω_w. After the spring, starting from position 1, has completed one turn in the time $T_s = 2\pi/\omega_{spin}$, the whirling motion will have displaced position 1 by an angle $\pm 2\pi\omega_w/\omega_{spin}$ (The + sign refers to the forward whirling and the − to the backward one). Therefore, in order to reach again the position 1 of maximum contraction, the spring takes a time T_{spring}, slightly different from T_s. We have $(T_{spring})^{-1} = (T_s)^{-1} \mp (T_w)^{-1}$. This means that each spring is forced to oscillate with amplitude $2r_w$ at the frequency $2\pi/T_{spring} = \omega_{spin} \mp \omega_w$ (The − sign is for the forward whirling and the + for the backward one). We see that these are the only frequencies at which the springs are forced to oscillate and that no deformations whatsoever take place in the spring's material at the whirling frequency. This is rather counterintuitive, since it is just the opposite of what happens in the more familiar case with $\omega_{spin} \ll \omega_\circ$, and since the two centers of mass O_1, O_2 of the two masses are seen (in the inertial reference frame) to rotate at $\omega_w \simeq \omega_\circ$ one with respect to the other. It would also be wrong to say that a deformation of the spring's material with frequency ω_w is superimposed to the one at ω_{spin}: the only effect of ω_w is to slightly correct ω_{spin} into $\omega_{spin} \mp \omega_w$. In the general case of an elliptical whirling motion (see Eqs. (1), (2), (3)) we have the superposition of the two dissipations at two angular frequencies $\omega_{spin} - \omega_w$ and $\omega_{spin} + \omega_w$ but still no dissipation at ω_w. These mechanical deformations are exactly the same as those of the one dimensional elastic oscillator of Fig. 2 if this oscillator has an identical spring (with stiffness $k/4$) and two small masses $\mu/2$ attached to its ends, with $\mu = k/(\omega_{spin} - \omega_w)^2$ so that its frequency of oscillation has the same value $\omega_{spin} - \omega_w$ as in the rotor of Fig. 1, and if it is made to oscillate with the same amplitude $2r_w$. The energy of the oscillator will decrease in time according to the law:

$$E_{oscillator}(t) = E_{oscillator}(0)e^{-(\omega_{spin}-\omega_w)t/Q_{sw}} \tag{24}$$

which defines its quality factor Q_{sw} and also defines the way by which Q_{sw} should be measured experimentally; Q_{sw} accounts for all losses in the oscillator at the frequency $\omega_{spin} - \omega_w$. The time derivative of (24) yields:

$$\dot{E}_{oscillator}(t) = -\frac{\omega_{spin} - \omega_w}{Q_{sw}} E_{oscillator}(t) \tag{25}$$

where $E_{oscillator} = (1/2)(k/4)(2r_w)^2 = kr_w^2/2 = m\omega_\circ^2 r_w^2/2$ is the energy of the oscillator in Fig. 2. Hence:

$$\dot{E}_{oscillator} = -\frac{\omega_{spin} - \omega_w}{Q_{sw}} \frac{1}{2} m\omega_\circ^2 r_w^2 \tag{26}$$

Since this oscillator has the same frequency and amplitude as each one of the 4 rotor's springs of Fig. 1, the total energy dissipated (as heat) inside the 4 rotor's springs is simply 4 times the energy dissipated by the oscillator of Fig. 2, that is:

$$\dot{E}_{rotor\,springs}(t) = 4\dot{E}_{oscillator}(t) = -\frac{\omega_{spin} - \omega_w}{Q_{sw}} 2m\omega_\circ^2 r_w^2 \tag{27}$$

which is nothing but the energy dissipated by the rotor because of <u>rotating damping</u>, i.e. because of friction between different parts of the rotor.

The conservation of energy requires that:

$$\dot{E}_{rotor}(t) + \dot{E}_w(t) = \dot{E}_{rotor\,springs}(t) \tag{28}$$

where:

$$E_{rotor} \simeq mR^2\omega_{spin}^2 \tag{29}$$

is the spin energy of the rotor and:

$$E_w = 2m\omega_w^2 r_w^2 \tag{30}$$

the energy (kinetic + elastic) of the whirling motion of the system. Eq. (28) says that the energy dissipated inside the rotor's springs cannot result only in a spin down of the rotor (i.e. the spin energy of the rotor cannot decrease by exactly the same amount as the energy dissipated inside the springs) because the conservation of angular momentum requires the development of a whirling motion which will gain angular momentum as well as energy, while the springs do not enter in the balance of angular momentum. From (29), using (21), and from (30), we have:

$$\dot{E}_{rotor} = -4mr_w\omega_w\omega_{spin}\dot{r}_w \quad , \quad \dot{E}_w = 4mr_w\omega_w^2\dot{r}_w \tag{31}$$

that is:

$$\dot{E}_w = -\frac{\omega_w}{\omega_{spin}}\dot{E}_{rotor} \tag{32}$$

which means that only the small fraction ω_w/ω_{spin} of the energy lost by the rotor is gained by the whirling motion, all the rest being dissipated as heat in the springs ($\dot{E}_{rotor\,springs} =$

$(1-\omega_w/\omega_{spin})\dot{E}_{rotor})$. From (28), using (27) for $\dot{E}_{rotor\,springs}(t)$ and (22) for $\dot{r}_w$ (needed to compute the time derivatives of (31)) we get:

$$\dot{\omega}_{spin} = -\frac{r_w^2}{R^2}\frac{\omega_\circ^2}{Q_{sw}}\frac{\omega_{spin}-\omega_w}{\omega_{spin}} + \frac{r_w^2}{R^2}\frac{\omega_w^2}{Q_w}\frac{\omega_w}{\omega_{spin}} \tag{33}$$

By substituting Eq. (23) into Eq. (33) we obtain:

$$Q_w = -Q_{sw}\frac{\omega_w^2}{\omega_\circ^2} \tag{34}$$

Clearly, the destabilizing (tangent) force along the whirling circle is (see (8) to (10)):

$$|F_{destab}| \simeq |\frac{1}{Q_w}F_{spring}| = \frac{1}{\frac{\omega_w^2}{\omega_\circ^2}Q_{sw}}|F_{spring}| \tag{35}$$

Apart for the case of very large viscous damping, Eq. (34) yields:

$$Q_w \simeq -Q_{spin} \tag{36}$$

because $\omega_w \simeq \omega_\circ \ll \omega_{spin}$, $Q_{sw} \simeq Q_{spin}$ (Q_{spin} accounts for all dissipation at the spin frequency). That is, the <u>negative</u> dissipation of the whirling (destabilizing) motion is equal and opposite to the dissipation of the rotor's springs when forced to oscillate at the spinning minus whirling frequency, which is essentially the <u>spinning</u> frequency. And Eq. (35), with (36), becomes:

$$|F_{destab}| \simeq \frac{1}{Q_{spin}}|F_{spring}| \tag{37}$$

(F_{destab} and F_{spring} are at about 90°). Hence,

$$|F_{destab}| \ll |F_{spring}| \qquad if \qquad Q_{spin} \gg 1 \tag{38}$$

This result applies to the GG experiment where the presence of a very large viscous damping ($Q_v \ll \omega_{spin}/\omega_\circ$) can certainly be ruled out. Indeed, with a planned spinning frequency of $5\,Hz$, the quality factor Q_{spin} of the springs (which accounts for <u>all</u> losses at the spin frequency) is certainly much larger than 1, and therefore the destabilizing forces are much smaller (by far) than the spring forces. An active damping force (servo force), opposite to F_{destab} and slightly larger will clearly stabilize the system.

In a rotating system dominated by a very large viscous damping it would be:

$$|F_{destab}| \simeq \frac{1}{\frac{\omega_w^2}{\omega_\circ^2} Q_v} |F_{spring}| \tag{39}$$

with $\omega_w \gg \omega_\circ$, $1 \leq Q_v \ll \omega_{spin}/\omega_\circ$ and $(\omega_w^2/\omega_\circ^2) Q_v \gg 1$: even in this case the destabilizing force generated by a very large amount of viscous friction is only a fraction of the spring force. It is therefore <u>not true</u> that: "*A minimal level of viscous damping has a serious effect on the performance estimate. The servo forces will dominate the passive spring forces*", as stated by the FPAG reviewing panel of ESA (resolution FPAG(96)4 of 9 October 1996, Point no. 2).

4. Stabilization with Rotating Active Dampers

Let the whirling motion be damped by electrostatic sensors/actuators fixed to the rotor. By providing forces internal to the system they cannot change its total angular momentum: they can only transfer the angular momentum of whirl to the rotation angular momentum of the rotor by spinning it up. This is what happens if they are made to provide a stabilizing force of the same intensity as the destabilizing one (37). This force must always act along the vector of relative velocity of the centres of mass of the bodies in their whirling motion, as seen in the inertial frame of reference. Since the centres of mass of the bodies are displaced by an amount $2r_w$, the electrostatic plates will necessarily apply also a small force tangent to the surface of the rotor amounting to a fraction $2r_w/R$ of the main component F_a of the active force, of intensity $F_a \simeq (2/Q_{spin})|F_{spring}|$ (for both bodies), which will damp the relative velocity of whirl. Of the corresponding reaction components on the electrostatic actuators only the reaction to the small tangent component $f_a \simeq (1/Q_{spin})|F_{spring}|(2r_w/R)$ will produce a non zero angular momentum by spinning up the rotor at the expense of exactly the angular momentum of whirl:

$$f_a R \simeq \frac{1}{Q_{spin}} |F_{spring}| 2r_w \simeq \frac{1}{Q_{spin}} 2m\omega_w^2 r_w^2 = \dot{L}_w \tag{40}$$

which will therefore increase the spin angular momentum of the rotor $L_{rotor} = 2mR^2\omega_{spin}$ in such a way that the total angular momentum of the system is conserved. That is:

$$2mR^2 \dot{\omega}_{spin} \simeq \frac{1}{Q_{spin}} 2m\omega_w^2 r_w^2 \tag{41}$$

thus producing a spin up of the rotor at the rate $\dot{\omega}_{spin} \simeq (1/Q_{spin})(r_w^2/R^2)\omega_w^2$. Except for the sign this is essentially the same as (23) because $Q_w \simeq -Q_{spin}$ and $Q_{spin} \simeq Q_{sw}$. By integrating $\dot{\omega}_{spin}$ for the entire duration of the mission $T_{mission} = t_f - t_i$, from intial to final epoch, the ratio ω_f/ω_i of final–to–initial spin angular velocity of the rotor is obtained:

$$\frac{\omega_f}{\omega_i} = 1 + \frac{\omega_i}{Q_{spin}} \frac{r_w^2}{R^2} \frac{\omega_w^2}{\omega_i^2} T_{mission} \tag{42}$$

In the GG case, even assuming a very low value for the quality factor at the spin frequency of $5\,Hz$ ($Q_{spin} \simeq 500$), $R \simeq 5\,cm$ (for the smallest test body made of Pt/Ir), $r_w \simeq 10^{-6}\,cm$ (the capacitance sensors and actuators that provide the active damping can in fact maintain the radius of whirl within a smaller value than this) and $\omega_w^2/\omega_{spin}^2 \simeq 2.5 \cdot 10^{-6}$ we get, for a 6 months duration of the mission:

$$\frac{\omega_f}{\omega_i} - 1 \simeq 10^{-13} \tag{43}$$

The corresponding angular advance is:

$$\Delta\theta \simeq \frac{1}{2Q_{spin}} \frac{r_w^2}{R^2} \omega_w^2 T_{mission}^2 \simeq 5\,arcsec \tag{45}$$

which is clearly negligible. The corresponding quality factor Q_{rotor} for the spin up of the rotor (in absence of any external disturbances to its spin rate) would be obviously huge:

$$Q_{rotor} = -2\pi \frac{T_{mission}}{T_{spin}} \frac{E_{rotor}}{\Delta E_{rotor}} \simeq -2.5 \cdot 10^{21} \tag{46}$$

the amounts of energy and angular momentum gained in 6 months being vanishingly small:

$$\Delta E_{rotor} \simeq 2 \cdot 10^{-13} E_{rotor} \simeq 2 \cdot 10^{-4}\,erg \quad , \quad \Delta L_{rotor} \simeq 10^{-13} L_{rotor} \simeq 6 \cdot 10^{-6}\,g\,cm^2\,s^{-1} \tag{47}$$

Let us now consider the conservation of energy in the presence of rotating active dampers. The contribution to the energy by the electrostatic dampers comes from the work done by both components of the active force. The larger one, of intensity F_a and the smaller one of intensity f_a:

$$F_a \simeq \frac{2}{Q_{spin}} 2m\omega_w^2 r_w^2 \quad , \quad f_a \simeq \frac{2}{Q_{spin}} 2m\omega_w^2 r_w^2 (\frac{r_w}{R}) \quad (r_w/R < 2 \cdot 10^{-7}) \tag{48}$$

While only f_a will transfer angular momentum from the whirling motion to the rotor, both components of the active force will provide energy. By spinning up the rotor at the rate $\dot{\omega}_{spin} \simeq (1/Q_{spin})(r_w^2/R^2)\omega_w^2$, f_a will obviously also increase the spin energy of the rotor by supplying to it the power:

$$\dot{E}_{rotor} \simeq \frac{\omega_{spin}}{Q_{spin}} 2m\omega_w^2 r_w^2 \tag{49}$$

As for the component F_a (always directed along the relative velocity vector of the centres of mass of the bodies and opposite to it) it will supply energy at the rate:

$$\dot{E}_{F\,a} \simeq \frac{\omega_{spin} - \omega_w}{Q_{spin}} 2m\omega_w^2 r_w^2 \tag{50}$$

because the electrostatic dampers, being fixed to the rotor (hence spinning at ω_{spin}), are required to provide a force at frequency ω_w (in order to damp the whirling motion at ω_w) and therefore must actuate at frequency $\omega_{spin} - \omega_w$; in order to give the required stabilizing force they must necessarily supply energy at the rate (50). This is transferred to the springs, to be dissipated as heat, while the energy of the whirling motion does not change any longer. Thus, the springs are provided with a power $(1/Q_{spin})(\omega_{spin} - \omega_w) \cdot 2m\omega_w^2 r_w^2$ which is exactly the energy that they dissipate in the presence of a whirling motion of constant radius r_w (see Eq. (27)).

In GG the electrostatic sensors/actuators are fixed to the spinning bodies and actuate at the frequency $\omega_{spin} - \omega_w$, i.e. close to the spin/signal frequency of $5\,Hz$. This means that they produce noise close to $5\,Hz$ (with respect to the fixed frame, i.e. close to $10\,Hz$ or DC w.r.t. the rotating frame). We demonstrate in Paper II that the weakly suspended PGB laboratory inside the GG spacecraft is very effective in attenuating vibrational perturbations which act at frequencies close to the spin/signal frequency with respect to the non rotating frame. This is very useful to attenuate noise produced by the electrostatic dampers, by FEEP thrusters (which also fire close to the spin/signal frequency), and in general by any other effect which may disturb the experiment at the spin/signal frequency. For the action of the electrostatic actuators to be successful it is necessary that the whirling circle be sufficiently small, its centre defining the position of relative equilibrium of the system given the original unbalance $\vec{\epsilon}$ of the rotor because of manufacturing and mounting errors and of its spin-to-natural frequency ratio. In GG this is achieved after initial unlocking by means of inch-worms equipped with pressure sensors. Once inch-worms have achieved the initial centring by removing the unbalance bias $\vec{\epsilon}$, the electrostatic dampers can damp the whirling motions and stabilize the system (by providing forces as small as we have calculated) also in the presence of external disturbances such as drag. All this has been confirmed with numerical simulations (performed by Alenia Spazio) assuming very low quality factors (20 for the suspensions of the PGB and 500 for those of the test masses) and including drag disturbances as well as implementation errors. Using forces of the right intensity, and allowing for removal of the initial unbalance, the work by Alenia Spazio shows that no problems arise due to the fact that the damping force is provided by spinning actuators. There is no indication whatsoever that the damping force needs to be amplifyed by a factor $\omega_{spin}/\omega_\circ$ if applied in the rotating frame as stated in the ESTEC Report.

Numerical simulations of the GG system do not yet include the small movable rods, pivoted at their centres on flat elastic gimbals, to which the GG test masses are suspended. As of our present understanding it is possible that, due to the differential oscillations of the bodies, they will develop only conical whirling motions, and no cylindrical whirling motion. If so, they will not require any active stabilization, as conical whirls are naturally damped. If a closer analysis will show that this is not the case, they can easily be stabilized similarly to all other bodies of the GG system. Obviously, in introducing them with a non-zero mass in the numerical simulations they will also be given the appropriate values for the principal moments of inertia which will best suit their stabilization. The possibility of adjusting the ratios of the principal moments of inertia in a body of cylindrical symmetry is well known.

Conservation of energy and angular momentum in the presence of spinning actuators definitely demonstrates that whirling motions in weakly coupled rotors (such as GG) can be stabilized by means of extremely small forces, far smaller than the spring forces and never competing with them. It should also be stressed that the fastness of the electronics is not an issue because, although the electrostatic dampers must actuate at a frequency close to the spin frequency, the whirling motion to be damped is much slower, so that corrections and adjustments are possible over several spin periods. Furthermore, this electronics is the same for all the GG bodies, and as a matter of fact it is also similar to the electronics needed for drag–free control with FEEP thrusters. In no way it can be regarded as a critical issue in the space experiment.

Far more important it is to stress the fact that the major source of dissipation in the GG experiment, namely the springs, dissipate energy at the frequency of spin, not at the frequency of whirl. This is true only because of the fast rotation as compared to the small natural frequency of oscillation provided by the weak mechanical coupling. This means that thermal noise will be dominated by the quality factor of the suspension at $5\,Hz$, not at the much smaller natural frequency. This Q is bound to be very high, making thermal noise small and the integration time short, in spite of the fact that the experiment is run at room temperature. The advantage of weak coupling and fast spin is apparent once more.

5. Stabilizing Forces in Ground Rotating Machines with $\omega_{spin} \gg \omega_\circ$

The result (37), which so far has been obtained on the basis of general physical principles, is the same used in engeneering textbooks and literature on rotating machines. On the basis of direct experience with many rotating machines (whose suspensions are certainly not the tiny GG springs) it is concluded that friction inside rotating parts (the suspensions) is essentially of structural nature, thus always obtaining a frequency of whirl very close to the natural frequency; see Eqs. (11) to (14). As a consequence, the coefficient of rotating damping (see e.g. G. Genta, *Vibration of Structures and Machines*, Springer 1993, Section 4.5.5) when $\omega_{spin} \gg \omega_\circ$ is given as:

$$c_r \simeq \eta \frac{k}{\omega_{spin}} \tag{51}$$

where $\eta \simeq 1/Q$ is the internal loss of the material at the frequency at which the material goes through the elastic hysteresis cycle, which is $\omega_{spin} - \omega_\circ \simeq \omega_{spin}$, and not $\omega_\circ$ as stated in the ESTEC Report. Thus:

$$\eta = \frac{1}{Q_{spin}} \tag{52}$$

and:

$$c_r = \frac{1}{Q_{spin}} \frac{k}{\omega_{spin}} \tag{53}$$

In order to stabilize the whirling motion which is known to develop because of the rotating damping expressed by the coefficient (53) it is necessary to provide an amount of non rotating damping, expressed by a coefficient c_{nr} which satisfies the stability condition of the rotor (known as "Jeffcott rotor"):

$$\frac{c_{nr}}{c_r} > \frac{\omega_{spin}}{\omega_\circ} \qquad\qquad (k = m\omega_\circ^2) \tag{54}$$

hence

$$c_{nr} > \frac{1}{Q_{spin}} m\omega_\circ \tag{55}$$

From this, the required stabilizing (damping) force can be computed, since the velocity to be damped (in the inertial reference frame) is —at any given time— the linear velocity of the centre of mass along the whirling circle of radius r_w:

$$F_{stabiliz} = c_{nr}\omega_\circ r_w > \frac{1}{Q_{spin}} m\omega_\circ^2 r_w = \frac{1}{Q_{spin}} k r_w = -\frac{1}{Q_{spin}} F_{spring} \tag{56}$$

$$|F_{stabiliz}| > \frac{1}{Q_{spin}} |F_{spring}| \tag{57}$$

If $\omega_\circ \ll \omega_{spin}$ then $Q_{\omega_{spin}-\omega_\circ} \simeq Q_{spin}$ so that Eqs. $(37), (38), (56), (57)$ remain almost exactly valid also in the general case of an elliptical whirling motion (see Eqs. $(1), (2), (3)$).

6. Comparison with the ESTEC Result

In none of these two different derivations we recover the result reported in the *Appendix* to the ESTEC Technical Assessment of GG (as released on October 7, 1996), namely:

$$\left[|F_{stabiliz}| > \frac{1}{Q} |F_{spring}| \frac{\omega_{spin}}{\omega_\circ} \right]_{ESTEC Appendix} \tag{58}$$

The ESTEC *Appendix* contains in fact an incorrect definition of the non–rotating damping forces which are needed to stabilize the whirling motions. On page 16 (lines 18–20) one can read: "... *non-rotating damping forces are obtained by virtue of the naturally unavoidable viscous friction between the rotating body and the non-rotating parts, ...*". This is in fact the definition of the friction in the bearings, <u>which is not</u> <u>the non–rotating damping</u> needed for stabilization (see below; see also the GG Blue preprint, §III). Also incorrect are the definitions, given on page 41 (beginning of Section 2.1) for the rotating and non–rotating damping. It is stated: "*Consider the general case where c_r represents the viscous damping coefficient in the rotating frame, and c_n represents the viscous damping coefficient in the inertial frame ('non–rotating damping')*". The damping coefficients derive from physical friction, hence from dissipated energy, which do not depend on the reference system from where they are looked at. The physical dimension of damping coefficients is *mass/time*, which in Galilean mechanics does not depend on the reference frame. The "rotating damping" and the "non–rotating damping" are not the same effect as seen from different reference frames: they are the <u>names</u> of <u>two different types</u> of damping in the <u>same</u> reference frame. They have different physical properties (respectively in destabilizing and in stabilizing the rotor) that <u>do not depend</u> on the reference frame. Moreover, as we have just said above, they have nothing to do with a third type of damping: the viscous friction in the bearings.

In Fig. 3 we show the <u>three</u> general types of friction in a rotating machine, from which the correct definition of rotating and non–rotating damping is obtained. They are:

i) The <u>friction in the bearings</u>. This is the friction (mostly viscous) between the rotating body and the non rotating parts, which is obviously effective in slowing down the rotor but almost completely ineffective at damping whirling motions (and also at producing them). An important advantage of the GG space experiment is the absence of bearings, hence of bearings friction at all (and we certainly don't want to simulate them actively!).

ii) The <u>rotating damping friction</u>. This is the friction (viscous plus structural) between two parts of the system which are <u>both rotating</u> (i.e. two parts of the rotor). The corresponding losses are those which produce the instabilities (whirling motions) in weakly suspended rotors with $\omega_\circ \ll \omega_{spin}$ by giving rise to the destabilizing forces computed above (namely $1/Q_{spin}$ of the spring elastic forces). In the GG space experiment where all rotors are suspended with tiny springs in vacuum the rotating damping friction is essentially structural, caused by the relative motion of the various parts of the springs subject to mechanical deformations (at spin minus natural frequency, which is essentially the spin frequency). Adding the rotating friction generated by the rotating electrostatic active dampers themselves, which provide a force much smaller than the spring force, does in no way change the dynamics of the GG system.

iii) The non–rotating damping friction which generates the non–rotating damping forces. This is the friction (viscous plus structural) between two parts of the system both non–rotating (i.e. between two parts of the non-rotating supports, for example the friction between the non-rotating part of the bearings and their fixed supports). The non-rotating damping forces are effective in damping transverse translational oscillations of the rotor's axis of rotation (for example the whirling motions), and they can do this without slowing down its rotation. For the whirling motions to be stabilized they must simply provide a coefficient of non–rotating damping which satisfies the inequality (55). For instance, in the ground rotors in which non–rotating damping is provided by tipping the non–rotating part of the bearing in oil this is essentially viscous damping. In the GG space experiment where there are no non–rotating parts an equivalent non–rotating damping is provided by electrostatic actuators fixed in the rotating system. (see §4)

From a physical viewpoint the most important characteristic distinguishing the effects of rotating and non–rotating damping on one side from the effects of friction in the bearings on the other is that the former produce forces on the rotor, while the latter produces torques. They are therefore independent from one another and interact only to a second order, namely because of construction errors, asymmetries, misalignements etc. From this it follows that if one were in fact using the forces generated by the friction in the bearings in order to stabilize the whirling motions (rather than the forces due to non rotating damping), he would inevitably need extremely large forces.

As for the amplifying factor $\omega_{spin}/\omega_\circ$ claimed in the ESTEC *Appendix* as due to the fact that the active dampers are fixed to the rotating bodies, we have shown in §4 above that there is no physical grounds for it. So, the results of the ESTEC *Appendix* are based on the use of supposedly "stabilizing" forces which are a factor $(\omega_{spin}/\omega_\circ)^2 \simeq 10^6$ larger than we have shown (both theoretically and with numerical simulations) to be sufficient for damping the GG whirling motions. Let us see what is the effect of the huge *ESTEC "stabilizing" force* when applied to one of the masses m undergoing a destabilizing forward whirling motion at frequency $\omega_w \simeq \omega_\circ$ (as demonstrated by Eqs. (11) to (14) for all cases except the one of very high viscous damping, which can be ruled out in GG) at distance r_w from the equilibrium position. Since the *ESTEC "stabilizing" force* amounts to $F_{ESTEC} \simeq (1/Q)m\omega_{spin}^2 r_w$ it is apparent that it is compatible with two effects. In one case it could force the body to whirl at angular velocity ω_{spin}, much larger than its previous angular velocity of whirl ω_w when the system was undamped, at a distance r_w/Q from the equilibrium position, or, it could maintain the angular velocity of whirl of the undamped situation while pushing the body a distance $(\omega_{spin}/\omega_\circ)^2 r_w/Q$ away from the equilibrium position. If $\omega_{spin} \simeq 10^3\omega_\circ$ and Q is rather small (e.g. 20 for the PGB and 500 for the test masses), then it is apparent that in either case the huge ESTEC force, far from damping the whirling motions would force the two masses into a totally wrong dynamical configuration overcoming the spring forces and thus disrupting the whole experiment.

Appendix — Frequency of Whirl Due to Structural and Viscous Rotating Damping

Consider the rotor of Fig. 1. The relative motion of the two masses is described (in the inertial reference frame) by the equation:

$$m_{red}\ddot{z} + c_r\dot{z} + (k - i\omega_{spin}c_r)z = 0 \tag{A1}$$

where z is the position vector in the complex plane, m_{red} is the reduced mass of the system, c_r is the coefficient of <u>rotating damping</u> and there is no <u>non-rotating damping</u> (see §6). The characteristic equation associated to the equation of motion of the system is:

$$-m_{red}\lambda^2 + ic_r\lambda + k - i\omega_{spin}c_r = 0 \tag{A2}$$

The solution of Eq. $(A2)$ is of the type $z = z_0\exp(i\lambda t)$, so that the real part of λ, $\Re\lambda$, is the angular frequency of the whirling motion when the imaginary part of λ is negative. Therefore the destabilizing forward whirling motion has frequency:

$$\omega_w \equiv +\Re\lambda = \frac{1}{\sqrt{2}}\cdot\sqrt{\omega_0^2 - \frac{c_r^2}{4m_{red}^2} + \sqrt{\left(\omega_0^2 - \frac{c_r^2}{4m_{red}^2}\right)^2 + \left(\frac{c_r\omega_{spin}}{m_{red}}\right)^2}} \tag{A3}$$

Let us compute $\Re\lambda$ for a system dominated by structural damping and for a system dominated by viscous damping.

- <u>Structural rotating damping</u>. This means that c_r is written as:

$$(c_r)_s = \frac{1}{Q}\cdot\frac{k}{\omega_{spin} - \omega_\circ} \tag{A4}$$

hence:

$$\omega_w \equiv +\Re\lambda = \frac{\omega_\circ}{\sqrt{2}}\cdot\sqrt{1 - \frac{\omega_\circ^2}{4Q^2(\omega_{spin}-\omega_\circ)^2} + \sqrt{\left(1 - \frac{\omega_\circ^2}{4Q^2(\omega_{spin}-\omega_\circ)^2}\right)^2 + \frac{\omega_{spin}^2}{Q^2(\omega_{spin}-\omega_\circ)^2}}} \tag{A5}$$

and, if $\omega_\circ/\omega_{spin} \ll 1$:

$$\omega_w \equiv +\Re\lambda = \frac{\omega_\circ}{\sqrt{2}}\cdot\sqrt{1 + \sqrt{1 + 1/Q^2}} = \frac{\omega_\circ}{\sqrt{2}}\cdot\sqrt{2 + 1/(2Q^2)} \tag{A6}$$

Since rotating damping derives from dissipation at frequency $\omega_{spin} - \omega_\circ \simeq \omega_{spin}$, the corresponding Q is certainly such that $Q \gg 1$, yielding:

$$\omega_w \equiv +\Re\lambda \simeq \omega_\circ\Big(1 + \frac{1}{8Q^2}\Big) \simeq \omega_\circ \tag{A7}$$

- Viscous rotating damping. This means that c_r is written as:

$$(c_r)_v = \frac{1}{Q_v} \cdot \frac{k}{\omega_\circ} \tag{A8}$$

with Q_v the quality factor of the system due to rotating friction of viscous nature. Then:

$$\omega_w \equiv +\Re\lambda = \frac{\omega_\circ}{\sqrt{2}} \cdot \sqrt{1 - \frac{1}{4Q_v^2} + \sqrt{\Big(1 - \frac{1}{4Q_v^2}\Big)^2 + \Big(\frac{\omega_{spin}}{Q_v\omega_\circ}\Big)^2}} \tag{A9}$$

Since $\omega_\circ/\omega_{spin} \ll 1$ and $Q_v > 1$, the quantity $\omega_{spin}^2/(Q_v\omega_\circ)^2$ is dominant with respect to $1/(2Q_v^2)$ and we have:

$$\omega_w \equiv +\Re\lambda = \frac{\omega_\circ}{\sqrt{2}} \cdot \sqrt{1 - \frac{1}{4Q_v^2} + \sqrt{1 + \Big(\frac{\omega_{spin}}{Q_v\omega_\circ}\Big)^2}} \tag{A10}$$

It follows that the whirling frequency is:

$$\begin{cases} \omega_w \equiv +\Re\lambda = \sqrt{\frac{\omega_{spin}\omega_\circ}{2Q_v}} \gg \omega_\circ & if \quad Q_v \ll \frac{\omega_{spin}}{\omega_\circ} \\ \omega_w \equiv +\Re\lambda = \omega_\circ\sqrt{\frac{1+\sqrt{2}}{2}} \simeq 1.1\omega_\circ & if \quad Q_v \simeq \frac{\omega_{spin}}{\omega_\circ} \\ \omega_w \equiv +\Re\lambda = \omega_\circ\Big(1 + \frac{\omega_{spin}^2}{8\omega_\circ^2 Q_v^2}\Big) \simeq \omega_\circ & if \quad Q_v \gg \frac{\omega_{spin}}{\omega_\circ} \end{cases} \tag{A11}$$

The case of very large viscous damping can certainly be ruled out in GG (and probably also in ground rotating machines with $\omega_{spin} > \omega_\circ$). This means that in GG the coefficient of viscous rotating damping ($A8$) can never be used with a value of Q_v such that $Q_v \ll \omega_{spin}/\omega_\circ$. We shall have $(c_r)_v = (1/q_v) \cdot k/\omega_{spin}$ ($Q_v = q_v\omega_{spin}/\omega_\circ$) with $q_v \gg 1$ in the case of small viscous friction, and $q_v \simeq 1$ in the intermediate case. Hence, $(c_r)_v = (c_r)_s \cdot Q/q_v$ where Q is the quality factor due to structural damping. This means $(c_r)_v \simeq (c_r)_s$ in the case of small viscous friction and $(c_r)_v \simeq (c_r)_s \cdot Q$ in the intermediate case. The force required to stabilize the whirling motion is in all cases smaller than the elastic force of the spring (Eqs. (37), (39)). In GG, rotating friction comes from dissipation, in vacuum and at the spin frequency, in the tiny suspension springs and in the small rotating electrostatic plates which provide stabilizing forces much smaller than the spring forces themselves (see §4). Therefore, rotating structural friction is bound to be very small and rotating viscous friction (e.g. due to imperfect clamping of the suspensions) very small, if any. Ground tests based on the measurement of $\omega_\circ$, ω_w and ω_{spin} will be performed to determine the nature of rotating damping in the system using the results $(A7)$ and $(A11)$.

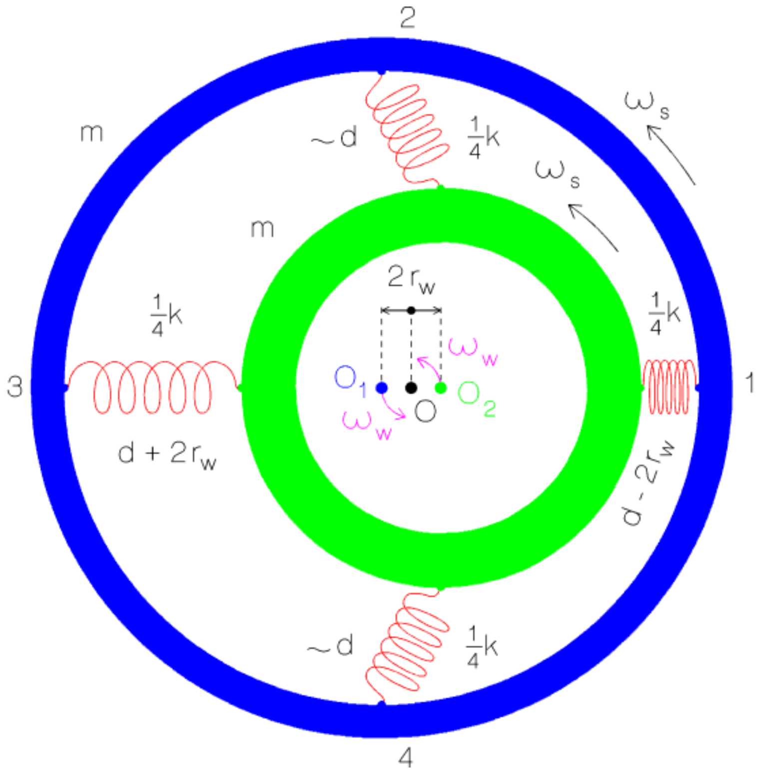

Figure 1. Section across the spin axis of the system of two hollow cylindrical masses coupled by weak springs. Both masses are spinning at the same angular velocity around their respective centers of mass O_1 and O_2. In their turn O_1 and O_2 are "whirling" around the center of mass O of the whole system, at a distance $r_w << R$ and at the angular velocity ω_w much smaller than the spin angular velocity.

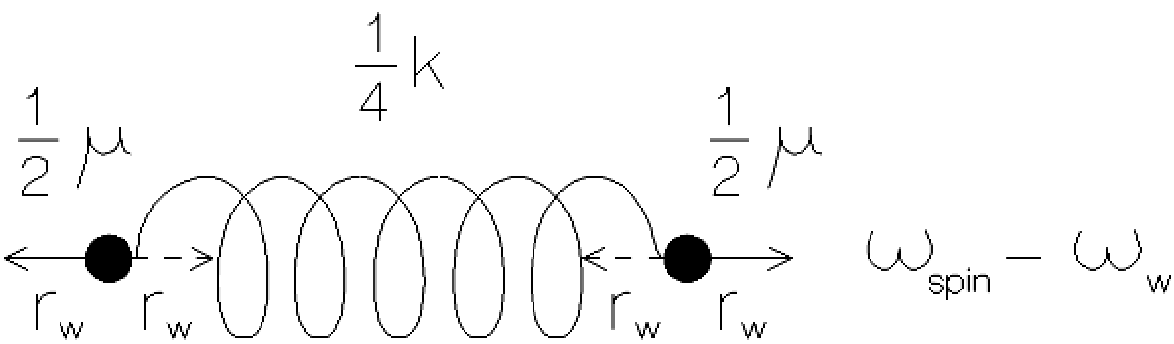

Figure 2. Simple scheme for the measurement of the quality factor of the spring at the frequency of spin minus the frequency of whirl.

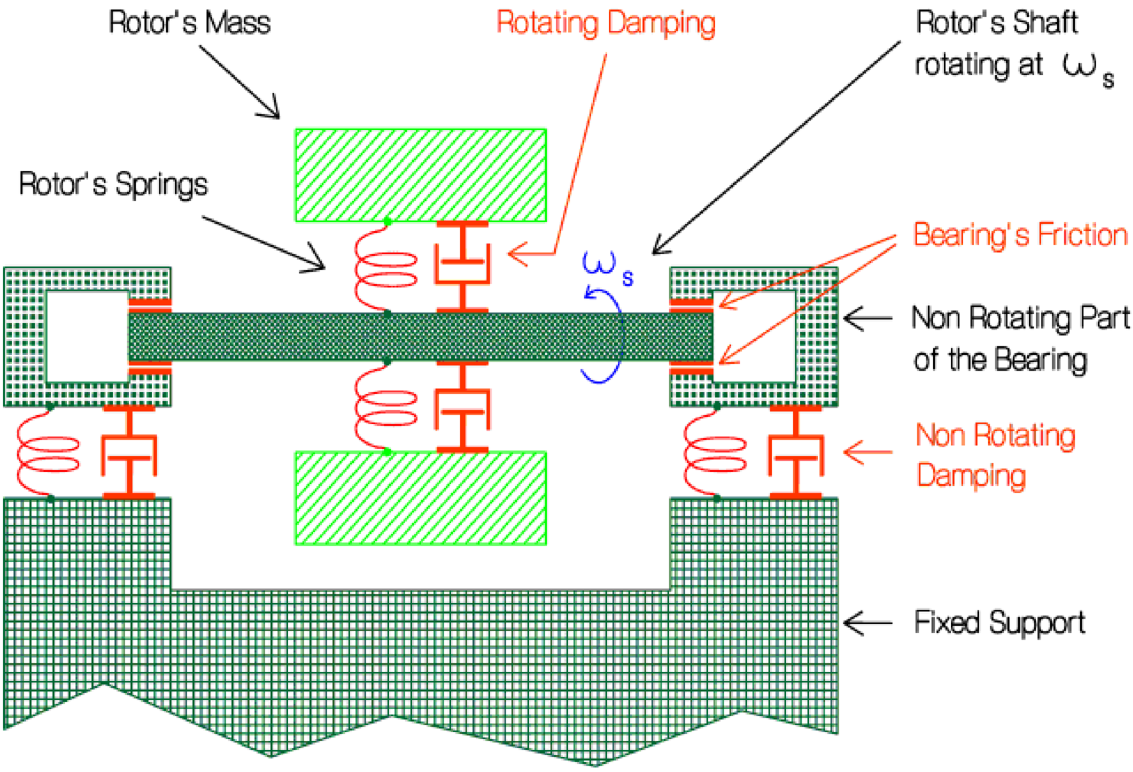

Figure 3. Rotating machine with rotating damping, non-rotating damping and friction in the bearings.