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RESEARCH ARTICLE

Limitations to testing the equivalence principle with satellite laser ranging

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Abstract We consider the possibility of testing the equivalence principle (EP) in the gravitational field of the Earth from the orbits of LAGEOS and LAGEOS II satellites, which are very accurately tracked from ground by laser ranging. The orbital elements that are affected by an EP violation and can be used to measure the corresponding dimensionless parameter η are semimajor axis and argument of pericenter. We show that the best result is obtained from the semimajor axis, and it is limited—with all available ranging data to LAGEOS and LAGEOS II—to $\eta \simeq 2 \times 10^{-9}$, more than 3 orders of magnitude worse than experimental results provided by torsion balances. The experiment is limited because of the non uniformity of the gravitational field of the Earth and the error in the measurement of semimajor axis, precisely in the same way as they limit the measurement of the product GM of the Earth. A better use of the pericenter of LAGEOS II can be made if the data are analyzed searching for a new Yukawa-like interaction with a distance scale of one Earth radius. It is found that the pericenter of LAGEOS II is 3 orders of magnitude more sensitive to a composition dependent new interaction with this particular scale than it is to a composition dependent effect expressed by the η parameter only. Nevertheless, the result is still a factor 500 worse than EP tests with torsion balances in the gravitational

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field of the Earth (i.e. at comparable distance), though a detailed data analysis has yet to be performed. While EP tests with satellite laser ranging are not competitive, laser ranging to the Moon has been able to provide a test of the EP almost 1 order of magnitude better than torsion balances. We show that this is due to the much greater distance of the test masses (the Earth and the Moon) from the primary body (the Sun) and the correspondingly smaller gradients of its gravity field. We therefore consider a similar new experiment involving the orbit of LAGEOS: testing LAGEOS and the Earth for an EP violation in the gravitational field of the Sun. We show that this test may be of interest, though it is a factor 300 less sensitive than in the case of the Moon due to the fact that LAGEOS is closer to the Earth than the Moon and consequently its orbit is less affected by the Sun. The limitations we have pointed out for laser ranging can be overcome by flying in low Earth orbit a spacecraft carrying concentric test masses of different composition with the capability, already demonstrated in ground laboratories, to accurately sense in situ any differential effects between them.

1 Introduction

The equivalence principle (EP) is usually tested by testing the Universality of Free Fall (UFF) which can be expressed by the dimensionless Eötvös parameter η defined as the *differential acceleration* Δa between two test bodies of different composition freely falling with acceleration “a” (usually referred to as *driving acceleration*) in the gravitational field of a given source mass:

$$\eta \equiv \frac{\Delta \mathrm{a}}{\mathrm{a}}. \tag{1}$$

If $\eta = 0$, UFF holds and there is no EP violation. From (1) it is apparent that the larger the driving acceleration, the more sensitive the EP test. Yet, the best laboratory results ($\eta \lesssim 10^{-12}$) have so far been obtained by the Eöt–Wash group in a remarkable series of precision experiments using test masses suspended on a rotating torsion balance in the gravitational field of the Earth [1] and of the Sun [2]. In these cases the driving acceleration is 1.7×10^{-2} m/s^2 and 6×10^{-3} m/s^2 at most, whereas it is 9.8 m/s^2 in Galileo-like mass dropping tests in the gravitational field of the Earth. This is because, in spite of the smaller driving signal, torsion balances are inherently very sensitive and, for EP tests, also differential instruments; in addition, signal modulation obtained by rotating the balance on a turntable up-converts the frequency of the signal, thus reducing $1/f$ noise.

One could think of exploiting the larger driving acceleration of the Earth on unsuspended test masses by putting them in low orbit around it, so as not to be limited by the short duration of the time of fall. LAGEOS satellites are the best existing example of test masses orbiting in the field of the Earth. They are designed as spherical, dense, cannon balls covered by corner cube reflectors to allow laser ranging from Earth. The area-to-mass ratio is minimized in order to reduce non gravitational perturbations. The orbiting altitude is not very low (about 1 Earth radius), in order to reduce drag effects from residual Earth atmosphere, but still the gain in terms of driving acceleration (in the field of the Earth) is a factor 400 with respect to torsion balance tests if they are performed in the field of the Sun and a factor 150 if they are performed in the field of the Earth.

In Sect. 2, we provide the correct theoretical framework for EP tests based on LAGEOS-like satellites, which is that of a 2-body gravitational problem treated in the general case in which the EP might be violated. The possibility of testing the EP with test masses freely orbiting around the Earth has been investigated in the recent past by several authors [3–5]. All previous results can be set in a common framework once the correct theoretical bases are given. These theoretical bases are used in Sect. 3 to establish how accurately the EP can be tested by measuring the effects of a possible violation on the size of the orbit and on the pericenter of LAGEOS-like satellites. The conclusion is that these LAGEOS-based EP tests are far from being competitive with the results provided by torsion balance experiments, in spite of the much stronger driving signal of the Earth and the very long integration time.

For completeness, in Sect. 4 we investigate also the effects on the orbit of LAGEOS-like satellites of an EP violation due to a new Yukawa-like interaction which, in addition to being composition dependent, would act at the particular distance scale of one Earth radius. In this 2-parameter framework we find that the effect of such a new interaction on the pericenter of LAGEOS II is 3 orders of magnitude bigger as compared to the effect on the same keplerian element of an EP violation based on the η parameter only. However, since the latter provides very poor results (as established in Sect. 3), it is found that an EP test based on a Yukawa-like interaction is also far from being competitive with torsion balance tests in the field of the Earth [1].

In Sect. 5 we briefly address Lunar Laser Ranging (LLR) versus Satellite Laser Ranging (SLR) as far as EP testing is concerned, to show how LLR has been able to achieve—for the Moon and the Earth freely falling in the gravitational field of the Sun—an EP test even slightly more sensitive than those obtained with torsion balances, namely to $\simeq 10^{-13}$. This naturally leads us to assess the sensitivity achievable in a new EP experiment using LAGEOS and the Earth in the gravitational field of the Sun, which turns out to be about 2 orders of magnitude worse than it is for the Moon and the Earth. Finally, we consider both SLR and LLR—from a conceptual point of view—versus space tests of the EP which can be carried out inside spacecraft such as the proposed GG, μSCOPE, STEP and GREAT [6–9]. We show the potentiality that these experiments have, by allowing the relative motion of the test masses to be very accurately sensed in situ, to exploit the stronger driving signal of the Earth in order to significantly improve the sensitivity of EP tests.

2 The 2-body gravitational problem with EP violation

Let us have a primary body of mass M (e.g. the Earth) and a secondary body of mass m (e.g. one LAGEOS satellite) with $m \ll M$, as in the case of artificial satellites of the Earth. The bodies are assumed to be point masses. To each body it is possible to ascribe a gravitational mass (the *gravitational charge*), indicated with subscript g, and an inertial mass, indicated with subscript i. Any deviation from 1 (more precisely, from $+1$) of the ratio gravitational-to-inertial mass will indicate an EP violation.

For the masses of the primary and secondary body we assume

$$M_g \equiv M_i(1+\eta_p), \quad m_g \equiv m_i(1+\eta_s). \tag{2}$$

If $\eta_s \neq 0$ the inertial and gravitational mass of the secondary body do not equate each other, and therefore it violates the EP; if $\eta_p \neq 0$, it would be the primary body (e.g. the Earth) to violate the EP, the respective values of η depending on the materials each body is made of. It should be noticed however that, in the case of the Earth, the self-gravitation binding energy is a relatively large fraction of its total mass-energy:

$$|f_\oplus| \simeq \frac{3GM^2/5R_\oplus}{Mc^2} \simeq 4.6 \times 10^{-10} \tag{3}$$

(with $R_\oplus$ the radius of the Earth and c the speed of light). For a very accurate EP test this value may not be negligible, as it is when both the test masses are artificial laboratory bodies, and will contribute to the test. The equation of motion of the secondary body, possibly depending on its composition, is

$$\ddot{\vec{r}} = -\frac{GM_i}{r^3}(1+\eta_p)(1+\eta_s)\vec{r} \tag{4}$$

which, since the product $\eta_p\,\eta_s$ is totally negligible, can be written as:

$$\ddot{\vec{r}} = -\frac{GM_i}{r^3}(1+\eta)\vec{r} \tag{5}$$

where

$$\eta = \eta_p + \eta_s, \tag{6}$$

thus involving the composition of the Earth (dominated by its iron-nickel core) and the composition of the satellite (an aluminum sphere with a brass core for both LAGEOS and LAGEOS II).

Note that torsion balance experiments are designed to measure the effect of a differential acceleration Δa (in the horizontal plane) which would arise if the test bodies on the balance were attracted by the Earth with different intensity because of their different composition. In this case, we have: $\Delta a/a = \eta_2 - \eta_1$, where η_2, η_1 for the two bodies are defined by $m_{2g} \equiv m_{2i}(1+\eta_2)$, $m_{1g} \equiv m_{1i}(1+\eta_1)$, and the products $\eta_p\,\eta_2$, $\eta_p\,\eta_1$ have been neglected. Similarly, EP tests to be performed with weakly coupled concentric test bodies of different composition placed inside a spacecraft in orbit around the Earth, would measure a relative differential acceleration equal to the difference $\eta_2 - \eta_1$ for the two bodies (again neglecting the products $\eta_p\,\eta_2$, $\eta_p\,\eta_1$). Instead, two LAGEOS satellites are not concentric and do not have the same initial conditions (in particular, they do not fly at the same altitude). Thus, each orbit is determined independently, from an equation of motion whose leading term at second hand member (to which in a real experiment a number of gravitational and non gravitational perturbations need to be added) is given by (5), with $\eta_p + \eta_{s_1}$ for one satellite and $\eta_p + \eta_{s_2}$ for the other, having neglected the products $\eta_p\,\eta_{s1}$, $\eta_p\,\eta_{s2}$ in each case. Two

LAGEOS satellites in orbit (or even more in the future) would allow us to combine the orbital elements that are relevant for EP testing (semimajor axis and argument of pericenter) so that some error source may be eliminated, but this is not a differential experiment and one should be careful, by writing the equations of motion properly, as to what kind of composition dependence the experiment is actually testing.

From (5) it is apparent that a value of η different from zero—i.e. an EP violation—would be equivalent to a change of the mass of the primary body (more precisely to a change of the product GM_i), while the force is still a central force *obeying the classical inverse square law*. Depending on the sign of η, it would be as if the central body were more or less massive than in the *classical* $\eta = 0$ case. As a result, an EP violation would rescale the satellite orbit, changing only its size. In this context, we find it appropriate to refer to the case $\eta \neq 0$ as *non galilean*, since it would violate the Universality of Free Fall—first tested by Galileo—hence the EP. Instead, if $\eta = 0$ we are back to the *classical* 2*-body gravitational problem of Celestial Mechanics* in which UFF and the EP hold, and we therefore refer to this case as *classical*.

Since the gravitational force is a central force, the orbital angular momentum (per unit of inertial mass m_i)

$$\vec{J} \equiv \vec{r} \times \dot{\vec{r}} \tag{7}$$

is conserved and therefore the motion is planar. Moreover, since the effect of an EP violation is a (small) variation of the intensity of the gravitational attraction, the direction of the angular momentum vector—i.e. the orbital plane and its inclination in space—are unaffected. The angular momentum modulus is

$$J = r^2\dot{\theta} \tag{8}$$

where (r, θ) are polar coordinates in the orbit plane. In these coordinates, using the angular momentum integral of motion to eliminate $\dot{\theta}$, we get the equation of motion in the r variable only (from now on we use simply M to mean M_i):

$$\ddot{r} = -\frac{GM}{r^2}(1+\eta) + \frac{J^2}{r^3} \tag{9}$$

We then introduce the Lenz vector in the general form

$$\vec{e} \equiv \frac{1}{GM(1+\eta)}\dot{\vec{r}} \times \vec{J} - \hat{e}_r \tag{10}$$

which is an integral of motion as in the *classical* $\eta = 0$ case, namley

$$\dot{\vec{e}} = \vec{0}\,. \tag{11}$$

Being an integral of motion, the Lenz vector provides a fixed direction in the orbital plane, which turns out to be the symmetry axis of the orbit. It is worth stressing that (11) holds only as long as gravitation obeys the inverse square law, which is the case with (5) where the deviation from the *classical* 2-body problem is only in the dependence on composition. Thus, the scalar product of the Lenz vector with the radial unit vector provides (like in the *classical* $\eta = 0$ case) the equation of the orbit in the (r, f) polar coordinates, f being the true anomaly of the secondary body, namely its angular position as measured from the (fixed) direction of the Lenz vector. The equation of the orbit so obtained is:

$$r = \frac{J^2/[GM(1+\eta)]}{1 + e\cos f}\,. \tag{12}$$

This is the equation of a conic (just as in the *classical* $\eta = 0$ case) of which the Lenz vector is the symmetry axis (see the dependence on $\cos f$), pointing towards the position of minimum distance from the primary (the pericenter), its modulus being the eccentricity of the conic. And the eccentricity is not affected by the value of η because only the size of the orbit is affected by it.

Note that also the direction of the Lenz vector in the orbit plane is unaffected by an EP violation. This must be the case because of symmetry reasons: since the direction of this vector is the direction of the symmetry axis of the orbit, any effect due to $\eta \neq 0$ must be the same on the two equal halves of the orbit. A direct proof that the direction of the Lenz vector is independent from η can be obtained starting from its definition (10) and then computing it at pericenter in the two cases $\eta = 0$ and $\eta \neq 0$ (being an integral of motion, we can choose any position along the orbit to compute it).

The linear size of the conic along its symmetry axis is

$$2a = r(f = 0) + r(f = \pi) \tag{13}$$

hence

$$J^2 = GM(1+\eta)a(1-e^2) \tag{14}$$

which gives the orbital angular momentum integral as function of the eccentricity e and semimajor axis a of the orbit. Note that we use the italic font "a", typically used in Celestial Mechanics to indicate the semimajor axis of the orbit, while the roman font "a" was used in (1) to indicate the acceleration of the free falling test body. As in the *classical* case, orbits are closed if $e = 0$ (circular) or $0 < e < 1$ (elliptic). By using (14) we can rewrite the equation of the orbit as:

$$r = \frac{a(1-e^2)}{1 + e\cos f} \tag{15}$$

and then in the form

$$\frac{r}{a} = \frac{1-e^2}{1 + e\cos f} \tag{16}$$

which is independent of η, thus expressing the fact that a value $\eta \neq 0$ would rescale the satellite orbit.

The energy function (per unit of inertial mass m_i) is

$$E = \frac{1}{2}\mathrm{v}^2 - \frac{GM(1+\eta)}{r}. \tag{17}$$

Since the system is isolated and there is no dissipation, the energy is an integral of motion whose value can be obtained by combining the value of the angular momentum given by (14) and the value of the Lenz vector, which is given by:

$$e^2 = 1 + \frac{2EJ^2}{(GM)^2(1+\eta)^2} \tag{18}$$

so that the energy integral is:

$$E = -\frac{GM(1+\eta)}{2a}. \tag{19}$$

For closed orbits the third law of Kepler is derived, as in the *classical* 2-body problem, starting from the orbital period P written as the ratio between the area of the ellipse $\pi a^2(1-e^2)^{1/2}$ and the velocity V_{area} (area of the ellipse scanned by the radial vector per unit time), which is in effect an integral of motion since it is half the angular momentum modulus:

$$V_{\text{area}} \equiv \frac{1}{2} r^2 \dot{f} = \frac{1}{2} J. \tag{20}$$

We therefore have, using (14) for the angular momentum integral:

$$P = \frac{2\pi a^2(1-e^2)^{1/2}}{J} = \frac{2\pi a^{3/2}}{(GM)^{1/2}(1+\eta)^{1/2}} \tag{21}$$

which is the third Kepler's law, usually written in terms of the mean motion $n = 2\pi/P$ (the average orbital angular velocity):

$$n^2 a^3 = GM(1+\eta). \tag{22}$$

In summary, the 2-body problem with EP violation is described by the same formulas derived in the *classical* 2-body problem, provided that whenever they contain the product GM, we substitute it with $GM(1+\eta)$. The result is a change in the orbit size. If $\eta = 0$ (i.e. no EP violation), all results naturally yield the corresponding expected results of the *classical* 2-body problem.

Note that the results of this Section are not limited to the case—assumed at the start of the Section—that the secondary mass be negligible compared to the mass of the primary. If the two masses are comparable, the equation of motion (5) becomes

$$\ddot{\vec{r}} = -\frac{GM_{\text{tot}}}{r^3}(1+\eta)\vec{r} \tag{23}$$

which is now written per unit of the (inertial) reduced mass $\mu_i = M_i m_i/(M_i + m_i)$ of the system and where $M_{\text{tot}} = M_i + m_i$ is the total inertial mass of the two bodies. As in the approximated case, the dimensionless parameter η expresses a dependence on the composition of the interacting bodies, including –if they are large celestial bodies—the contribution from their self-gravitation binding energies. This is the equation of motion of the general 2-body problem (with EP violation) once it has been reduced—with no loss of generality—to the motion of a single body, whose mass is the reduced mass of the system, orbiting around a fixed body, of mass equal to the total mass of the two bodies, separated by the relative position vector at the relative velocity vector of the real bodies. All formulas then follow from (23), just like formulas (7) to (22), simply by replacing the inertial mass M of the primary with the total inertial mass M_{tot}, and remembering that they are now written per unit of reduced (not secondary) mass. If the secondary mass is negligible, the reduced mass is the secondary mass, the total mass is the mass of the primary and we are back to the previous case.

3 Testing the EP with LAGEOS-like satellites

Within the theoretical framework developed in Sect. 2 we can now address the problem of using LAGEOS-like satellites for testing the EP. The problem is formulated in the following way.

Let us consider one LAGEOS satellite, and ask the following question: is it possible to establish from its orbital motion a deviation from zero in the value of the dimensionless η parameter as it appears in the equation of motion (5), hence, an EP violation? And if yes, to what accuracy?

In the motion of LAGEOS around the Earth, like in the motion of planets around the Sun and of natural satellites around their own planet, the best measured physical quantity is the orbital period. We can therefore assume the orbital period of LAGEOS to be known exactly. For all its other physical quantities we shall write a relationship between each quantity in the presence of composition dependence (UFF and EP violation to the level $\eta \neq 0$, subscript *ng* standing for *non galilean* as introduced at the beginning of Sect. 2), and the corresponding quantity in the *classical*, no EP violation case ($\eta = 0$, subscript c). Whenever needed, we shall expand these relationships in terms of η, and since we know from torsion balance experiments that η must be smaller than 10^{-12} we shall retain only terms to first order in η.

We already know that the unit vectors of the orbital angular momentum and the Lenz vectors $\hat{J}$ and $\hat{e}$ (i.e., the orbit plane and the symmetry axis of the orbit) as well as the eccentricity of the orbit cannot be used to measure η because they are not affected by a possible EP violation

Let us then start from the third Kepler's law (22) and write it in the $\eta \neq 0$ and $\eta = 0$ case, for the same (exactly known) mean motion $n = 2\pi/P$. By equating the orbital period in the two cases we obtain the relationship between the semimajor axis of the orbit in the *non galilean* $\eta \neq 0$ case and its value in the *classical*, $\eta = 0$, one:

$$a_{\mathrm{ng}} \simeq a_c \left(1 + \frac{1}{3}\eta\right). \tag{24}$$

This immediately gives, for the energy integral (19):

$$E_{\mathrm{ng}} \simeq E_c \left(1 + \frac{2}{3}\eta\right) \tag{25}$$

and for the angular momentum integral (since the eccentricity is independent from η):

$$J_{\mathrm{ng}} \simeq J_c \left(1 + \frac{2}{3}\eta\right) \tag{26}$$

The same relationship must hold, because of (20), for the velocity at which the elliptic orbit is scanned by the radius vector:

$$V_{\mathrm{area-ng}} \simeq V_{\mathrm{area-}c} \left(1 + \frac{2}{3}\eta\right) \tag{27}$$

The area of the ellipse, because of (24), depends on η just like this velocity:

$$\mathrm{Area}_{\mathrm{ng}} \simeq \mathrm{Area}_c \left(1 + \frac{2}{3}\eta\right) \tag{28}$$

and from this fact it follows that the orbital angular velocity is independent from η:

$$\dot{f}_{\mathrm{ng}} = \dot{f}_c \tag{29}$$

which is consistent with the fact we stressed at the beginning of Sect. 2, namely that the ratio r/a must be independent of η. In fact:

$$r_{\mathrm{ng}}^2 \equiv \frac{J_{\mathrm{ng}}}{\dot{f}_{\mathrm{ng}}} \simeq r_c^2 \left(1 + \frac{2}{3}\eta\right) \tag{30}$$

yielding, for the modulus of the radius vector:

$$r_{\mathrm{ng}} \simeq r_c \left(1 + \frac{1}{3}\eta\right) \tag{31}$$

and, in combination with (24)

$$\left(\frac{r}{a}\right)_{\mathrm{ng}} = \left(\frac{r}{a}\right)_c \quad . \tag{32}$$

Note that, since the ratio of the satellite radial distance from the primary at any time and the major semiaxis of its orbit is independent of η, no position along the orbit has any special physical meaning, not even the initial one, though it has been very much emphasized in the previous literature on the subject.

The effect of an EP violation, as expressed by the dimensionless parameter η, on the various physical quantities which characterize a LAGEOS satellite with orbital period P, can be summarized as follows.

The physical quantities which are independent from η (hence, do not allow it to be measured) are: $\hat{J}$ (the orbit plane); $\hat{e}$ (the symmetry axis of the orbit) and e (the eccentricity); the angular velocity along the orbit; the ratio r/a at any time along the orbit.

The physical quantities which depend on η to first order (and would in principle allow it to be measured) are: the major and minor semiaxes (in the same way), in fact, the modulus of the radius vector at any time; the energy and the orbital angular momentum, per unit of inertial mass of the satellite (in the same way); the area of the elliptical orbit and the velocity at which it is scanned by the radius vector (in the same way).

We now define:

$$\Delta a_{\mathrm{EP}} \equiv a_{\mathrm{ng}} - a_c \tag{33}$$

as the difference in semimajor axis between the value it would have in case of an EP violation to the level η and its value in the *classical* $\eta = 0$ case. With this definition, by means of (24), we can quantitatively relate the dimensionless parameter of EP violation η to the fractional difference in the semimajor axis of the satellite orbit caused by such violation, namely:

$$\eta \simeq 3\frac{\Delta a_{\mathrm{EP}}}{a_c} \tag{34}$$

and by means of this relationship, we can re-write all the quantities which turned out to depend on η in terms of $\Delta a_{\mathrm{EP}}/a_c$ instead:

$$E_{\mathrm{ng}} \simeq E_c\left(1 + 2\frac{\Delta a_{\mathrm{EP}}}{a_c}\right) \tag{35}$$

$$J_{\mathrm{ng}} \simeq J_c\left(1 + 2\frac{\Delta a_{\mathrm{EP}}}{a_c}\right) \tag{36}$$

$$r_{\mathrm{ng}} \simeq r_c\left(1 + \frac{\Delta a_{\mathrm{EP}}}{a_c}\right) \tag{37}$$

$$\mathrm{Area}_{\mathrm{ng}} \simeq \mathrm{Area}_c\left(1 + 2\frac{\Delta a_{\mathrm{EP}}}{a_c}\right) \tag{38}$$

$$V_{\mathrm{area}-ng} \simeq V_{\mathrm{area}-c}\left(1 + 2\frac{\Delta a_{\mathrm{EP}}}{a_c}\right). \tag{39}$$

Thus, any difference between the *classical* and the *non galileain* case is a difference in energy, angular momentum modulus and orbit size, area of the elliptic orbit and velocity at which it is scanned by the radius vector, and such a difference is of the order of $\Delta a_{\mathrm{EP}}/a_c$.

However, it is obvious that, while Δa_{EP}—according to our definition (33)—is due to an EP violation, it could simply be due to measurement errors in the semimajor axis in the *classical* case with $\eta = 0$. In the latter case, from the third law of Kepler in the *classical* form (i.e. as in (22) with $\eta = 0$) and the assumption that the mean motion is perfectly known, while the semimajor axis is known with a measurement error Δa_{meas}, we have

$$\frac{\Delta(GM)}{GM} = 3\frac{\Delta a_{\text{meas}}}{a_c} \tag{40}$$

which expresses the well known fact that an error in the measurement of the semimajor axis of the satellite around the Earth determines the level to which the product GM of the Earth is known. However, if we do not know a priori whether the satellite satisfies the EP or not, it is apparent from (34) and (40) that such a satellite would allow us to test the EP only to the level

$$\eta_{\text{min}} \simeq 3\frac{\Delta a_{\text{meas}}}{a_c} \tag{41}$$

Δa_{meas} being the error made in the measurement of its semimajor axis. Values of η smaller than this will be undetectable.

Note that having two LAGEOS-like satellites of different composition around the Earth would not help because, in spite of the fact that the central body is the same, the uncertainty in GM remains: each LAGEOS provides *independently* a value for it which is affected by the error in the determination of its own semimajor axis, an error which is similar for the two satellites.

The dimensionless factor $\eta_{\text{min}} \simeq 3\Delta a_{\text{meas}}/a_c$ therefore sets the limit to which LAGEOS-like laser tracked satellites can be used to test the EP. To put it another way, since the effect of an EP violation is to change the size of the satellite orbit, there is no way to detect such a change as long as it is within the error with which the size of the satellite orbit is measured.

Satellite Laser Ranging (SLR) presently allows the major semiaxis of LAGEOS to be recovered to about 1 cm. Thus, we expect EP tests with LAGEOS to be limited to

$$\eta_{\text{minSLR}} \simeq 3\frac{10^{-2}\text{m}}{1.23 \times 10^{7}\text{m}} \simeq 2.4 \times 10^{-9}. \tag{42}$$

If we now use (40) and (41) as a way of obtaining a limit to the sensitivity in testing the EP with LAGEOS, and take the official values for GM and $\Delta(GM)$ given by the *International Earth Rotation and Reference System Services* ([10], p. 12) based on data from both existing LAGEOS satellites, we get:

$$GM = 3.986004418 \times 10^{14}\ \text{m}^3\text{s}^{-2} \tag{43}$$

$$\Delta(GM) = 8 \times 10^{5}\ \text{m}^3\text{s}^{-2} \tag{44}$$

hence

$$\frac{\Delta(GM)}{GM} \simeq 2 \times 10^{-9}. \tag{45}$$

This is the best determination of η achieved with all laser ranging data available so far to LAGEOS and LAGEOS II, a result by far not competitive with EP tests performed in the lab using torsion balances, which have systematically found no violation to 10^{-12}.

More in detail, single shot ranging data collected at the various SLR stations around the world are compressed in a format agreed within the community to provide the so called "normal points" (see e.g. [11]), currently accurate to $1 \div 3$ mm. The orbital elements of LAGEOS satellites are then determined by best fit to the normal points of orbital arcs whose individual duration is typically 15 days; individual arcs of longer duration are not suitable because the accumulated errors due to the difficulty in accurately modelling non gravitational perturbations would degrade the orbital elements. At present, any such orbital arc allows the semimajor axis of LAGEOS to be determined to about $1 \div 2$ cm. With long integration times many orbital arcs are available and the statistics improves. The orbital period of LAGEOS being about 3.8 h, in 15 days it performs about 94 orbits around the Earth; with a number N_{arcs} of such 15 d orbital arcs, we expect to reduce the error in semimajor axis, hence to improve the sensitivity in EP testing with $\sqrt{N_{\mathrm{arcs}}}$. Since, as we have shown above, an EP violation would affect the size of the orbit, i.e. both the major and minor semiaxis, by taking both of them into account in the data analysis we expect to improve the accuracy in determining the orbital parameters by a further factor $\sqrt{2}$. Yet, if we would like to achieve with LAGEOS-like satellites a sensitivity in EP testing of 10^{-12} —which ground based torsion balances have already achieved— starting from the current state given by (45), we would need an integration time of about 120, 000 years, which obviously rules out such an experiment.

However, one could think of performing a different EP test, based on the observation of a physical quantity which, unlike the semimajor axis, shows a secular variation, also affected by an EP violation.

It is known that in the presence of an oblate primary body, i.e. in the presence of a primary with a non zero quadrupole coefficient

$$J_2 = \frac{C - A}{M R_\oplus^2} \tag{46}$$

where C is the moment of inertia of the primary with respect to its rotation/symmetry axis, A its moment of inertia relative to any axis in its equatorial plane ($C > A$ for an oblate spheroid like the Earth), and $R_\oplus$ the equatorial radius (larger than the polar one), an artificial satellite orbiting around it with major semiaxis a, eccentricity e and inclination I will have a long term motion of its pericenter axis (the Lenz vector). The accumulation of this effect with time might allow many years of laser ranging data to be better exploited. In this case one should use the pericenter of LAGEOS II and not that of LAGEOS since the latter is poorly determined because of the small orbital eccentricity ($e_L \simeq 0.0045$, $e_{LII} \simeq 0.014$).

It can be shown that, in the general case with $\eta \neq 0$, the rate of change of the argument of the pericenter is:

$$\dot{\omega} \simeq -\frac{3}{4} \frac{(GM(1+\eta))^{1/2}}{a^{7/2}} J_2 R_\oplus^2 \frac{1 - 5\cos^2 I}{(1 - e^2)^2} \tag{47}$$

and the relationship between the *classical* and the *non galilean*, $\eta \neq 0$ rate of change turns out to be (using (24) for the semimajor axis, and the fact that the eccentricity and the orbital plane—hence the inclination—are all unrelated to η):

$$\dot{\omega}_{ng} \simeq \dot{\omega}_c \left(1 - \frac{2}{3}\eta\right) \tag{48}$$

where $\dot{\omega}_c$ is obtained from (47) with $\eta = 0$ and a_c for the semimajor axis. Thus, the effect of $\eta \neq 0$ on the rate of change of the argument of pericenter can be written as a fraction of $\dot{\omega}_c$ (in modulus):

$$\dot{\omega}_\eta \simeq \frac{2}{3}\eta\,\dot{\omega}_c. \tag{49}$$

Making the favorable assumption that only the error in J_2 of the Earth contributes to the error in the measurement of $\dot{\omega}_c$, the quantity which competes with $\dot{\omega}_\eta$ given by (49), from which we want to measure η, is

$$\Delta\dot{\omega}_{J_2} \simeq \dot{\omega}_c \frac{\Delta J_2}{J_2}. \tag{50}$$

By equating (50) to (49) we obtain the minimum value of η that can be measured with this experiment due to the error in the measurement of J_2 of the Earth. This value is:

$$\eta_{J_2} \simeq \frac{3}{2}\frac{\Delta J_2}{J_2} \tag{51}$$

It is interesting to note that, while the relative error $\Delta(GM)/GM$ limits the best EP test with LAGEOS based on the measurement of the size of its orbit, the relative error $\Delta J_2/J_2$ limits the best EP test that the measurement of the pericenter of (an eccentric) LAGEOS can provide. Thus, the two largest multipole moments of the Earth (the monopole and the quadrupole) are physical quantities deeply related to any deviation from the EP that we can try to extract from the orbit of a LAGEOS-like satellite around the planet.

The current best determination of J_2 is given by [12] and amounts to $\Delta J_2/J_2 \simeq 10^{-7}$ (obtained using the calibrated values and not the formal errors, as discussed by the authors), from which we get

$$\eta_{J_2} \simeq 1.5 \times 10^{-7} \tag{52}$$

which is 2 orders of magnitude worse than the value (45) obtained from observation of the orbit size.

However, if another LAGEOS satellite is launched, with rather large eccentricity so that its pericenter can be observed and combined with that of LAGEOS II, then, since J_2 of the Earth (as well as its equatorial radius) are the same for both orbits, by combining the rates of change of the two satellites one can construct an observable physical quantity (depending on the composition of the Earth and the two satellites) where J_2 has been eliminated and therefore the error in J_2 is no longer relevant. Then, the next relevant sources of error would be the error in J_4 of the Earth (the next zonal harmonic to affect the pericenter of the satellites) and the error Δa_{meas} in the measurement of the semimajor axis of each satellite (which appears along with the orbital inclination and eccentricity) in the *classical* rate of change of the argument of the pericenter $\dot{\omega}_c$.

The contribution from J_4 to the rate of change of the pericenter of LAGEOS II ($a_{LII} \simeq 12,163\,\mathrm{km}$, $e_{LII} \simeq 0.014$, $I_{LII} \simeq 52.65^o$) as calculated by [13] is:

$$\dot{\omega}_{J_4} \simeq 6 \times 10^{-5} J_4 \ \text{ rad/s} \tag{53}$$

while the contribution from J_2 (computed from (47) with $\eta = 0$) is

$$\dot{\omega}_{J_2} \simeq 8.2 \times 10^{-5} J_2 \ \text{ rad/s.} \tag{54}$$

Since $J_2 \simeq 10^{-3}$ and $J_4 \simeq 1.6 \times 10^{-6}$ [12], the contribution from J_2 is about 3 orders of magnitude bigger than the contribution from J_4 ($\dot{\omega}_{J_2} \simeq 8.2 \times 10^{-8}$ rad/s; $\dot{\omega}_{J_4} \simeq 9.6 \times 10^{-11}$ rad/s). However, once the error due to J_2 has been eliminated, the error due to J_4 must be taken into account. This reads

$$\Delta\dot{\omega}_{J_4} \simeq \dot{\omega}_{J_4} \frac{\Delta J_4}{J_4} \tag{55}$$

where $\Delta J_4/J_4 \simeq 7.3 \times 10^{-6}$ according to the best determination of the geopotential [12]. By equating (55) to (49) we obtain the minimum value of η that can be measured with this experiment due to the error in the determination of J_4, namely

$$\eta_{J_4} \simeq \frac{3}{2} \frac{\dot{\omega}_{J_4}}{\dot{\omega}_{J_2}} \frac{\Delta J_4}{J_4} \simeq 1.28 \times 10^{-8}. \tag{56}$$

As for the contribution from the error in the measurement of the semimajor axes of the satellites, we obtain it by expressing $\dot{\omega}_c$ (with the contribution from J_2 alone) as

$$\dot{\omega}_c \simeq -\frac{3}{4} n \left(\frac{R_\mathrm{e}}{a_c}\right)^2 J_2 \frac{1 - 5\cos^2 I}{(1 - e^2)^2} \tag{57}$$

from which a measurement error Δa_{meas} in semimajor axis yields:

$$\Delta\dot{\omega}_a \simeq -2 \frac{\Delta a_{\mathrm{meas}}}{a_c} \dot{\omega}_c. \tag{58}$$

By equating it to (49) (in modulus) we obtain the minimum value of η that can be measured because of the error in measuring the semimajor axis, namely,

$$\eta_{\mathrm{min}} \simeq 3 \frac{\Delta a_{\mathrm{meas}}}{a_c} \tag{59}$$

which is the same as (41) and therefore leads to the same limiting value of 2×10^{-9}. Since this value is 1 order of magnitude smaller than η_{J_4} computed above, we must conclude that an EP test based on the observation of the pericenters of LAGEOS II and of a future eccentric LAGEOS-like satellite, would be limited by the error in J_4 of the Earth to about $\eta \simeq 10^{-8}$, which is 4 orders of magnitude worse than EP tests already achieved, since many years, by torsion balances. As for the present time, with LAGEOS II only, an EP test based on its pericenter is limited to 10^{-7} see (52), a value even worse by 1 order of magnitude.

In summary, the best test of the EP achievable with the LAGEOS satellites is obtained from the measurement of the orbit size and we have shown that the corresponding value of the parameter η is the same as the relative error in the measurement of GM of the Earth, currently $\simeq 2 \times 10^{-9}$. EP tests performed with torsion balances are more than 3 orders of magnitude better than that. A different EP test based on the observation of the pericenter of LAGEOS II would be limited by the relative error in J_2 of the Earth, which is 2 orders of magnitude worse than the error in GM, thus yielding also an EP test 2 orders of magnitude worse. If the pericenter of LAGEOS II were combined with the pericenter of another eccentric LAGEOS yet to be launched, J_2 could be eliminated and the limiting source of error would then come from the zonal harmonic coefficient J_4 of the geopotential, resulting in an EP test to $\simeq 1.25 \times 10^{-8}$, only 1 order of magnitude better than using the pericenter of LAGEOS II alone and 4 orders of magnitude worse than torsion balance tests.

4 Test of a new Yukawa-like interaction with LAGEOS II

So far our analysis has referred to the case in which the EP that is at the foundation of General Relativity is tested by searching for a deviation from the Universality of Free Fall as function of one parameter only (η) in the $1/r$ potential of their mutual gravitational attraction. The existence of a new interaction has been suggested, expressed by a Yukawa-like potential (see e.g. [14])

$$V_{Yu}(r) = -\alpha \frac{GM_a M_b}{r} e^{-r/\lambda} \tag{60}$$

(M_a and M_b the masses of the interacting bodies) with a distance scale given by the parameter λ and proportional to a dimensionless parameter α. In the presence of this new interaction, a small satellite orbiting around the Earth of mass M would be subjected to an additional radial force (per unit of its inertial mass)

$$\Re_{Yu}(r) = -\alpha \frac{GM}{r^2} \left(1 + \frac{r}{\lambda}\right) e^{-r/\lambda}. \tag{61}$$

In the general case α may or may not depend on the composition of the interacting bodies (i.e. the Earth and the satellite). In the first case—which is the one relevant to our current analysis—a non zero value of α would indicate an EP violation. However, unlike EP violation expressed by the parameter η that we have analyzed in the previous Sections, the force (61), often referred to as "*5th force*", would affect a satellite only if its orbit size matches the particular distance scale λ of this new interaction. In other words, a given satellite is sensitive only to one particular value of λ, which for LAGEOS is 1 Earth radius (about half its semimajor axis).

The additional perturbing force (61) is still a central force, hence the satellite motion is still confined to a plane. However, it has a dependence on the mutual distance of the interacting bodies which deviates from $1/r^2$. In this case, as noticed in Sect. 2, Eq. (11) no longer holds, i.e. the Lenz vector is no longer an integral of motion. As a result, the pericenter direction is no longer fixed in the orbit plane and we therefore expect a non zero rate of change of the argument of pericenter even in the presence of a point-like or spherically symmetric central body (see [15] and [16]). The Lenz vector not being an integral of motion, the orbit equation in the form of a conic (12) can no longer be derived.

Since any expected deviation is extremely small, we can compute the dynamical effects of the new force with the perturbative methods of Celestial Mechanics, by writing the variational equations of the osculating orbital elements, in particular the semimajor axis and the argument of pericenter. Since the perturbation (61) acts only in the radial direction, it is convenient to write the variational equations in the form of Gauss (see e.g. [17], Chap. 3.2), which yield

$$\dot{a}_{Yu} = e\frac{2}{n\sqrt{1-e^2}}\Re_{Yu}(r)\sin f \tag{62}$$

and

$$\dot{\omega}_{Yu} = -\frac{\sqrt{1-e^2}}{ena}\Re_{Yu}(r)\cos f \tag{63}$$

where the orbital elements appearing at the second-hand member are those of the osculating orbit of the classical 2-body gravitational problem, and the time dependent relative distance r in $\Re_{Yu}(r)$ is expressed by the conic equation of the osculating (unperturbed) orbit:

$$r = \frac{a(1-e^2)}{1+e\cos f}. \tag{64}$$

It is then very important to establish whether the time varying effects (62) and (63) caused by the new Yukawa-like interaction have a non zero secular component or not. In order to compute the secular effects in semimajor axis and the argument of pericenter one has to average $\dot{a}_{Yu}$ and $\dot{\omega}_{Yu}$ over the orbital period, hence over the true anomaly f appearing as $\sin f$, $\cos f$ in (62) and (63) and also through r, as given by (64). This can be done by expanding $\sin f$ and $\cos f$ as Fourier series in the mean anomaly M (an angle measured from the pericenter, like f, whose rate of change is the mean motion n) with coefficients containing higher powers of the eccentricity for the higher harmonics (see e.g. [18], Chap. IV) and then averaging over the mean anomaly. The result obtained by [16] for $\dot{\omega}_{Yu}$ is

$$\dot{\omega}_{Yu} \simeq n\frac{\alpha}{4}e^{-a/\lambda}\left[\frac{1}{e}I_1 - \frac{\alpha}{\lambda}\left(1-\frac{e}{2}\right)I_0 - \frac{\alpha}{2\lambda}(e-1)I_1\cdots\right] \tag{65}$$

where

$$I_n = \frac{2}{\pi} \int_0^{2\pi} \cos(nM) e^{(ae/\lambda)\cos M} dM \tag{66}$$

are the Bessel functions in integral form. For $\dot{\omega}_{Yu}$ their contribution is different from zero, which means that there is a non zero secular variation of the rate of change of the pericenter. Instead, a similar calculation for $\dot{a}_{Yu}$ yields zero, namely, no secular variation in semimajor axis arises due to a new interaction such as (61).

For LAGEOS II the secular effect (65) peaks at a distance of about $1 R_\oplus$, while falling off rapidly for any such interaction acting on a scale λ larger or smaller than that. The accuracy to which the pericenter of LAGEOS II can be determined will constrain the parameter α of a new interaction acting at the distance scale of $1 R_\oplus$, and if α is composition dependent this would be a test of the EP at 1 Earth radius. The relevance of such a test must therefore be assessed by comparing the value of α which can be determined in this way (depending on the composition of the Earth and the satellite) with the experimental results obtained for the Eötvös parameter η by means of rotating torsion balances in the gravitational field of the Earth [1], yielding $\eta \simeq 10^{-12}$.

According to [16] and further calculations by one of us (D.M.L), a numerical evaluation of (65) for the orbit of LAGEOS II yields a peak value at $\lambda \simeq 1 R_\oplus$ amounting to

$$\dot{\omega}_{YuLII} \simeq 1.3 \times 10^{-4}\,\alpha \ \ \text{rad/s}. \tag{67}$$

Though this effect would be there even if the Earth were point-like and the satellite orbit a perfect ellipse, in reality, since the Earth is neither point-like nor spherically symmetric, the pericenter of LAGEOS II undergoes a secular rate of change also in absence of any new Yukawa interaction, the largest contribution coming from J_2 of the Earth (see (57)). In order to compare (67) with the effect that would be produced on the pericenter of LAGEOS II by an EP violation expressed by the parameter η (see (49)) it is worth comparing the following quantities:

$$\frac{\dot{\omega}_{YuLII}}{\dot{\omega}_c} \simeq 1.7 \times 10^3\,\alpha, \quad \frac{\dot{\omega}_\eta}{\dot{\omega}_c} \simeq 0.7\,\eta. \tag{68}$$

The comparison shows that, once normalized to the *classical* pericenter rate of change due to J_2 of the Earth, the effect of a Yukawa-like interaction on the pericenter of LAGEOS II (provided it has the distance scale of 1 Earth radius) would be 3 orders of magnitude bigger than the effect of an EP violation given by the η parameter. Thus, having considered the effect on the pericenter of a new interaction depending on two parameters rather than 1, makes it possible to improve considerably the potentiality of the test over the very poor result (52).

For a quantitative assessment, let us first assume that the only source of error in $\dot{\omega}_c$ is J_2. In this case we get an error $\Delta\dot{\omega}_c \simeq \dot{\omega}_c\,\Delta J_2/J_2 \simeq 8.2 \times 10^{-15}$ rad/s (with $\Delta J_2/J_2 \simeq 10^{-7}$ as given by [12]) to compete with (67) for the determination of α.

By imposing that they equal each other we get

$$\alpha_{\min} \simeq 6.3 \times 10^{-11}, \tag{69}$$

namely, the minimum value of α that could in principle be measured from the rate of change of the pericenter of LAGEOS II.

This limit has been calculated assuming that only J_2 contributes to the *classical* rate of change of the pericenter. In fact, all zonal harmonics of the Earth do, and a number of non gravitational forces contribute as well. A detailed error budget, including both gravitational and non gravitational effects, has been computed by [16]. The author concludes that seven years laser data of LAGEOS II would provide

$$\alpha_{7yr} \simeq 4.9 \times 10^{-10} \tag{70}$$

which is almost 1 order of magnitude worse than the estimate (69) based on the effect of J_2 alone, and only a factor 4 better than the value $\eta \simeq 2 \times 10^{-9}$ derived from the orbit size determined from all LAGEOS and LAGEOS II data (Sect. 3, (45)).

Note that we have assumed J_2 as determined independently with a given error. However, J_2 competes directly with the new interaction in affecting the pericenter, and it is itself determined from LAGEOS data, so it is a circular argument to take it as given while determining α, because an error in J_2 can be *absorbed* by the parameter α and viceversa. A thorough analysis of LAGEOS II laser ranging data, including α and λ as solve-for parameters in addition to all usual parameters of gravitational and non gravitational effects, has yet to be performed, to provide a more reliable value of α. However, even the current estimate (70) of α is about a factor 500 worse than the results already available for η from torsion balances in the gravitational field of the Earth [1]. We see no possibility that having in the future another more eccentric LAGEOS in orbit could significantly change this situation.

Instead, the pericenter of LAGEOS II can provide the best constraint on a new Yukawa-like interaction with 1 Earth radius range. A systematic analysis of all real data available is worth carrying out in order to reliably establish such constraints.

5 Concluding remarks

It is apparent from the work reported here that an experiment to test the EP with LAGEOS-like satellites is by far not competitive with torsion balance tests.

The best EP test with LAGEOS satellites is obtained from the measurement of the semimajor axis, and the level achieved is the same as the relative error in the product GM of the Earth, namely 2×10^{-9} (as obtained from laser ranging data to LAGEOS and LAGEOS II). Using the pericenter of the satellite orbit in the hope that the secular effect of an EP violation would help in measuring it, does indeed provide a result 2 orders of magnitude worse, unless another eccentric LAGEOS were launched in the future, in which case this result would improve but only by 1 order of magnitude. A better use of the pericenter is made if the data are analyzed searching for a new Yukawa-like interaction with a distance scale of 1 Earth radius. Error budget calculations performed for the pericenter of LAGEOS II indicate that, as an EP test, the result is only a factor 4 better than that provided by the semimajor axes of LAGEOS and LAGEOS II, and it is not competitive (by a factor 500) with torsion balance tests in the gravitational field of the Earth (i.e. at the same distance). However, the pericenter of LAGEOS II allows the new interaction to be constrained more stringently than in the past and a careful analysis of the actual LAGEOS II laser ranging data is needed to provide more realistic results than the estimates reported here on this issue.

These being the facts, the question arises as to how it is possible that, contrary to SLR, Lunar Laser Ranging (LLR) has allowed scientists to test the EP for the Earth and the Moon falling in the gravitational field of the Sun to $\simeq 10^{-13}$ using only about three decades of laser ranging data [19]. This question can be answered by looking at the minimum detectable value of η, given by (41), which should now be written in the case of LLR. While laser ranging to the Moon is at present almost as good as laser ranging to LAGEOS, the orbital distance of the free falling bodies (the Earth and the Moon) from the primary (the Sun) is no longer about 2 Earth radii but instead 1 AU. As a consequence, taking about 1 cm measurement error, we have

$$\eta_{\mathrm{minLLR}} \simeq 3\frac{10^{-2}m}{1.5\times 10^{11}m} \simeq 2\times 10^{-13} \tag{71}$$

thus showing no contradiction between the discouraging results obtained in Sect. 3 for LAGEOS and the remarkable achievements of Lunar Laser Ranging in testing the EP. Note that in this experiment the η being tested is the difference between the Earth and the Moon, thus depending on their difference in composition (as well as in self gravitation binding energy due to the large masses involved).

It is natural at this point to consider testing the EP with a new experiment, involving the Earth and one LAGEOS-like satellite in the gravitational field of the Sun: if LAGEOS and the Earth differ in composition and EP is violated, the satellite orbit around the Earth will be *polarized* along the Earth–Sun direction, towards the Sun or away from it depending on the sign of η. The orbit polarization phenomenon has been widely investigated for the orbit of the Moon ([20–23]). Within a first order perturbation analysis of the lunar orbit (assumed circular and planar), in the presence of an EP violation $\eta \neq 0$, the amplitude of polarization of its orbit is (see e.g. [23]):

$$|\delta r_p| = \frac{1+\frac{2n_{\mathrm{sat}}}{n_{\mathrm{sat}}-n_{\oplus}}}{n_{\mathrm{sat}}^2-(n_{\mathrm{sat}}-n_{\oplus})^2} n_{\oplus}^2 d_{\oplus\odot}\eta \tag{72}$$

where $d_{\oplus\odot}$ is the Earth–Sun distance, $n_{\oplus}$ is the mean angular velocity (mean motion) of the Earth around the Sun and $n_{\mathrm{sat}} - n_{\oplus}$ is the synodic mean motion of the satellite—the Moon in this case—around the Earth (n_{sat} being its sidereal mean motion). This equation can be rewritten in terms of the dimensionless parameter

$$m = \frac{n_{\oplus}}{n_{\mathrm{sat}}-n_{\oplus}} \tag{73}$$

in the form

$$|\delta r_p| = \frac{3}{2} m \frac{1 + \frac{2}{3}m}{1 + \frac{1}{2}m} d_{\oplus\odot}\eta. \tag{74}$$

Since m is a small parameter ($m \simeq 1/12.5687$ for the Moon, and much smaller for LAGEOS whose orbital period around the Earth is only $\simeq 3.8$ h), Eq. (74) can be expanded in powers of m to become:

$$|\delta r_p| = \frac{3}{2} m \left[1 + \frac{1}{6}m - \frac{1}{12}m^2 + \cdots \right] d_{\oplus\odot}\eta \tag{75}$$

In the case of the Moon the contribution from terms to order higher than 1 in m has been found (by neglecting second order terms in the lunar and solar eccentricities) to increase the effect to first order in m by a factor 1.62201 (see [23]). In the case of LAGEOS, $m \ll 1$ and second order terms can be neglected. The ratio between the polarization of the orbit of LAGEOS and the polarization of the orbit of the Moon is therefore, for the same value of η:

$$\frac{|\delta r_{\mathrm{pLAGEOS}}|}{|\delta r_{\mathrm{pMoon}}|} \simeq \frac{n_{\mathrm{Moon}} - n_{\oplus}}{n_{\mathrm{LAGEOS}}} \frac{1}{1.6} \simeq \frac{1}{300} \tag{76}$$

meaning that an EP violation of the same extent as for the Moon would require to detect, in the case of LAGEOS, a polarization of its orbit about 300 times smaller than the polarization of the lunar orbit. This is due to the fact that LAGEOS is much closer to the Earth than the Moon (by about a factor 30, because it is devoted to geodynamics studies) and therefore its orbit is much less affected by the Sun than the orbit of the Moon.

Since SLR data allow the orbit of LAGEOS to be recovered to an accuracy comparable to (or somewhat better than) the accuracy to which the orbit of the Moon is recovered from LLR data, we conclude from (76) that it should be possible to test the EP between LAGEOS and the Earth, freely falling in the gravitational field of the Sun, to about 10^{-11}. Though this result would not be competitive as compared to either EP tests with LLR or to torsion balance tests, it would in fact be of interest for both LLR and SLR scientific communities. The LLR community can test the methods developed for the Moon on each LAGEOS (and other such satellites), in which case—though the technology of laser tracking is similar—the orbit and physical model are very different. For instance, non gravitational effects are more relevant for LAGEOS than for the Moon, while lunar librations and tidal evolution effects are not there for LAGEOS, and any contribution to EP violation from its self gravitational energy is negligible. It is like conducting an experiment—previously limited to the only existing natural satellite of our planet—with a substantially different apparatus characterized by different systematics. On the other hand, the SLR community has the possibility to test the physical model, parameters and data analysis methods developed and used for over three decades, by searching also for an EP violation, since it is expected that they should find no evidence for such an effect to about 10^{-11}.

Finally, it is worth going back to (41) and (71), since they raise an important issue: while an improvement is possible in EP tests with LLR by improving the technology of laser ranging to the Moon (hence, by reducing the measurement error which appears at the numerator on the right hand side of (71)) no breakthrough can reasonably be expected. Gaining one order of magnitude in laser ranging accuracy requires considerable efforts. The real ultimate limitation—particularly to SLR—as far as EP testing is concerned, is the fact that measurements are performed from Earth, in which case 1 mm accuracy of laser ranging is a great achievement.

The alternative is obvious: EP tests require to measure *differential accelerations* (see definition (1) of the Eötvös parameter η)—and the displacements they give rise to—of the test masses *relative* to one another. The semimajor axes of their individual orbits around the Earth are not relevant for testing the EP. Thus, if the test masses are placed inside a spacecraft and *differential* measurements are performed in situ with apparata similar to those used in the lab to detect relative displacements of macroscopic bodies, the accuracy achievable is many orders of magnitude better than by laser ranging. This is why proposed space experiments such as GG, STEP, μSCOPE, GReAT can in fact aim at far more accurate tests of the EP which are unthinkable of with SLR and beyond laser ranging technology even for LLR.

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ERRATUM

Limitations to testing the equivalence principle with satellite laser ranging

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Unfortunately, an error occurred in Eq. (65). The correct version is given here:

$$\dot{\omega}_{Yu} \simeq n\frac{\alpha}{4}e^{-a/\lambda} \cdot \left[\frac{1}{e}I_1 - \frac{a}{\lambda}\left(1-\frac{e}{2}\right)I_0 - \frac{a}{2\lambda}(e-1)I_1 \cdots\right]$$

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