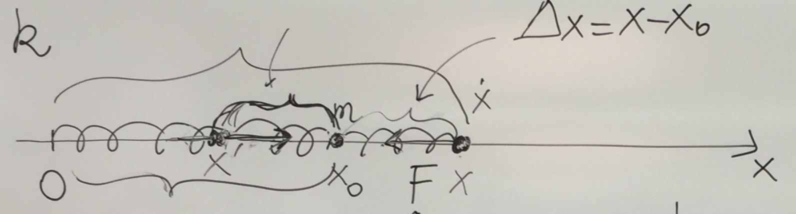
S6 - Lezioni 5 e 7 Aprile 2016:

- Soluzione del compitino del 17 Marzo, consegna e discussione
- Oscillatore armonico smorzato
- Introduzione e uso dei numeri complessi e degli esponenziali immaginari

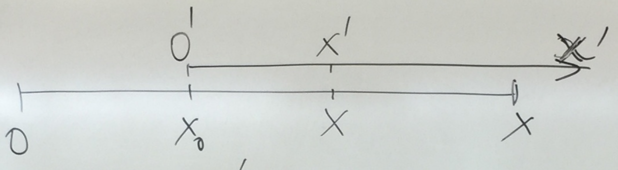
COMPITINO 17 MARZO 2016

k



$m \qquad x_0 > 0$

$k > 0$



$x' = x - x_0$

1. $F(x) = -k(x - x_0)$

$\Delta x \equiv x - x_0 \Rightarrow F(x) = -k\,\Delta x$

$\cancel{F(x) = -kx}$

2. $F(x') = -k\underbrace{(x' - x_0)}_{<0} > 0 = k\,\Delta x' > 0$

$\Delta x' = x_0 - x'$

$\Delta x' = \Delta x$

$\Downarrow$

$|F(x')| = |F(x)|$

3. $F(x)$ conservativa

$$U(P) - U(P_{rif}) \equiv -\mathcal{L}_{P_{rif} \to P}$$

$$U(x) - U(x_{rif}) = -\mathcal{L}_{x_{rif} \to x} = -\int_{x_{rif}}^{x} F(x')\,dx'$$

$$x_{rif} = x_0 \Rightarrow U(x_0) = 0$$

$$U(x) = -\int_{x_0}^{x} F(x')\,dx' = -\int_{x_0}^{x} (-k(x' - x_0))\,dx'$$

$$= k\int_{x_0 \to 0}^{x \to x - x_0} (x' - x_0)\,d(x' - x_0) = k\left[\frac{(x' - x_0)^2}{2}\right]_0^{x - x_0} = k\frac{(x - x_0)^2}{2} > 0$$

$$\int x\,dx = \frac{x^2}{2}$$

$$U(x)=k\int_{x_0}^{x}(x'-x_0)\,dx'=k\int_{x_0}^{x}x'\,dx'-kx_0\int_{x_0}^{x}dx'=$$

$$=\frac{k}{2}\left[x'^2\right]_{x_0}^{x}-kx_0\left[x'\right]_{x_0}^{x}=\frac{1}{2}k\left(x^2-x_0^2\right)-kx_0(x-x_0)$$

$$=\frac{1}{2}k\left[x^2-x_0^2-2x_0x+2x_0^2\right]=\frac{1}{2}k\left(x^2+x_0^2-2x_0x\right)$$

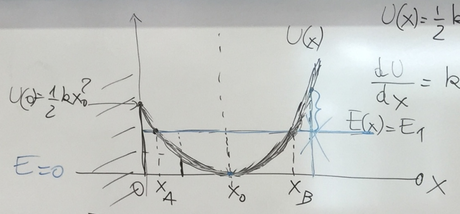
$$=\frac{1}{2}k(x-x_0)^2$$

4. $x, \dot{x}$ $\quad E(x,\dot{x})=\frac{1}{2}m\dot{x}^2+\frac{1}{2}k(x-x_0)^2=E_1$ costante

5. v_0 in $x_0 \xrightarrow{?} (\Delta x)_{max}$

$$E_1 = \frac{1}{2}m v_0^2 = \frac{1}{2}k(\Delta x)^2_{max} \Longrightarrow (\Delta x)_{max}=\frac{v_0}{\left(\sqrt{k/m}\right)=\omega_0}$$

6.



$U(x)=\frac{1}{2}k(x-x_0)^2$

$\frac{dU}{dx}=k(x-x_0)$

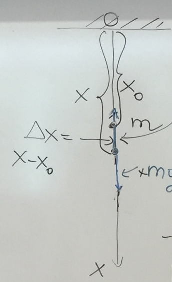
$\left.\frac{dU}{dx}\right|_{x=x_0}=0$

$E(x)=E_1>0$

in x_A: $U(x_A)=E_1$ $\quad U(x_B)=E_1$

$\Rightarrow v(x_A)=v(x_B)=0$

7. k?

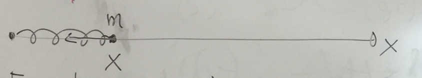


$-k\Delta x+mg=0$

$$k=\frac{mg}{\Delta x}=\frac{10\cdot 10^{-3}\cdot 9.8}{10^{-2}}=9.8\ \mathrm{N/m}$$

k, m OSCILLATORE ARMONICO SMORZATO

$m\ddot{x}+\beta\dot{x}+kx=0$



$F=-kx \qquad k>0$

$[\beta]=[N\,m^{-1}\,s] \qquad \beta>0$

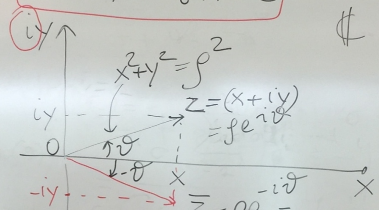
$= kg\,m\,s^{-2}\,m^{-1}\,s = kg\,s^{-1}$

$\ddot{x}+\frac{\beta}{m}\dot{x}+\frac{k}{m}x=0$

$\omega_0=\sqrt{\frac{k}{m}}$

$\frac{\beta}{m}=\lambda \qquad [\lambda]=[s^{-1}]$

$\boxed{\ddot{x}+\lambda\dot{x}+\omega_0^2x=0}$



$\bar{z}=\rho e^{-i\vartheta}=\rho(\cos\vartheta-i\sin\vartheta)$

$i\equiv\sqrt{-1}$

$i^2=-1$

$z=x+iy \in \mathbb{C}$

$\bar{z}=x-iy \qquad x,y\in\mathbb{R}$

$z\bar{z}=(x+iy)(x-iy)=x^2+\cancel{ixy}-i^2y^2-\cancel{ixy}=x^2+y^2\in\mathbb{R}$

$\mathbb{C}$ $\quad x=\rho\cos\vartheta$

$y=\rho\sin\vartheta$

$z=\rho\underbrace{(\cos\vartheta+i\sin\vartheta)}_{=e^{i\vartheta}}$

$\bar{z}=\rho(\cos\vartheta-i\sin\vartheta)$

$z\bar{z}=\rho^2\cos^2\vartheta+\rho^2\sin^2\vartheta$

$=\rho^2\underbrace{(\cos^2\vartheta+\sin^2\vartheta)}_{1}=\rho^2$

$$\boxed{e^{i\vartheta} \equiv \cos\vartheta + i\sin\vartheta}$$

$e = 2.718282$

$$z = \rho(\cos\vartheta + i\sin\vartheta) = \rho e^{i\vartheta}$$

ln

$$\cos\vartheta = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2}$$

$$\sin\vartheta = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$$

$$x^\alpha \cdot x^\beta = x^{\alpha+\beta}$$

$$\frac{x^\alpha}{x^\beta} = x^\alpha \cdot x^{-\beta} = x^{\alpha-\beta}$$

$$(x^\alpha)^\beta = x^{\alpha\beta}$$

$$x^0 = 1$$

log

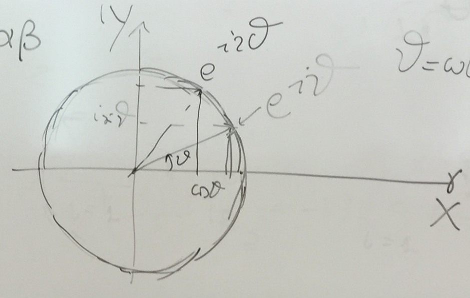
$$f(x) = e^x$$

$$\frac{de^x}{dx} = e^x$$

$$\frac{d}{dx} e^{(\alpha x)} = \alpha e^{\alpha x}$$

$z = 1 \cdot e^{i\omega t}$

$\vartheta = \omega t$



$z = x + iy \rightarrow (\ddot{z} + \lambda\dot{z} + \omega_0^2 z = 0)$ $\quad (\ddot{x} + \lambda\dot{x} + \omega_0^2 x) + i(\ddot{y} + \lambda\dot{y} + \omega_0^2 y) = 0$

? $\ddot{x} + i\ddot{y}$ $\quad \dot{z} + i\dot{y}$ $\quad x + iy$

$$\varsigma(t) = \alpha e^{\sigma t} \qquad \sigma \in \mathbb{C} \qquad \alpha \in \mathbb{C}$$

α e σ costanti $\neq 0$

$$\ddot{\varsigma} + \lambda\dot{\varsigma} + \omega_0^2 \varsigma = 0 \qquad \dot{\varsigma}(t) = \alpha\sigma e^{\sigma t} \qquad \ddot{\varsigma}(t) = \alpha\sigma\left(\frac{d\sigma t}{dt}\right)e^{\sigma t} = \alpha\sigma^2 e^{\sigma t}$$

$$\alpha\sigma^2 e^{\sigma t} + \lambda\alpha\sigma e^{\sigma t} + \omega_0^2 \alpha e^{\sigma t} = 0 \rightarrow (\sigma^2 + \lambda\sigma + \omega_0^2)\alpha e^{\sigma t} = 0$$

$\rightarrow \boxed{\sigma^2 + \lambda\sigma + \omega_0^2 = 0}$ eq. algebrica di 2° grado nella variabile σ

$$\sigma_\pm = -\frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} - \omega_0^2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \qquad \gamma = \frac{\lambda}{2} \in \mathbb{R}, > 0$$

$ax^2 + bx + c = 0$

$x_{1,2} = -\frac{b}{2} \pm \sqrt{b^2 - 4ac}$

DISCRIMINANTE

1. $\gamma^2 - \omega_0^2 < 0 \quad \omega_0^2 - \gamma^2 > 0 \quad \sqrt{\omega_0^2 - \gamma^2} = \omega \leq \omega_0$

$$\sigma_\pm = -\gamma \pm \sqrt{-1(\omega_0^2 - \gamma^2)} = -\gamma \pm i\omega \Rightarrow \varsigma(t) = \alpha e^{(-\gamma \pm i\omega)t}$$

$\xi(t) = \alpha e^{-\gamma t} e^{\pm i\omega t}$

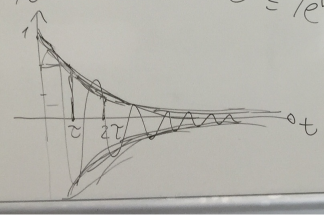
$e^{-\gamma t}$: $t=0$ $e^{0}=1$

$t/(1/\gamma)$ $t = 1/\gamma = \tau$ $e^{-1} = 1/e$

$\equiv \tau$ $t = 2\tau$ $e^{-2} = 1/e^{2}$

t/τ $\vdots$

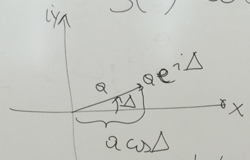
$e^{-t/\tau}$ $t = n\tau$ $e^{-n} = 1/e^{n}$



$\xi(0) = \alpha = a e^{i\Delta}$ $a, \Delta \in \mathbb{R}$

$\xi(t) = a e^{i\Delta} e^{-\gamma t} e^{\pm i\omega t} = a e^{-\gamma t} e^{i(\pm\omega t + \Delta)}$

$t=0$ $\xi(0) = a e^{i\Delta} = a(\cos\Delta + i\sin\Delta)$



caso 1. oscillatore sottosmorzato

$\gamma < \omega_0$

$\xi(t) = a e^{-\gamma t}\left(\cos(\pm\omega t + \Delta) + i\sin(\omega t + \Delta)\right)$

$\boxed{x(t) = a e^{-\gamma t}\cos(\pm\omega t + \Delta)}$ se $\Delta = 0$ $x(t) = a e^{-\gamma t}\cos(\omega t)$