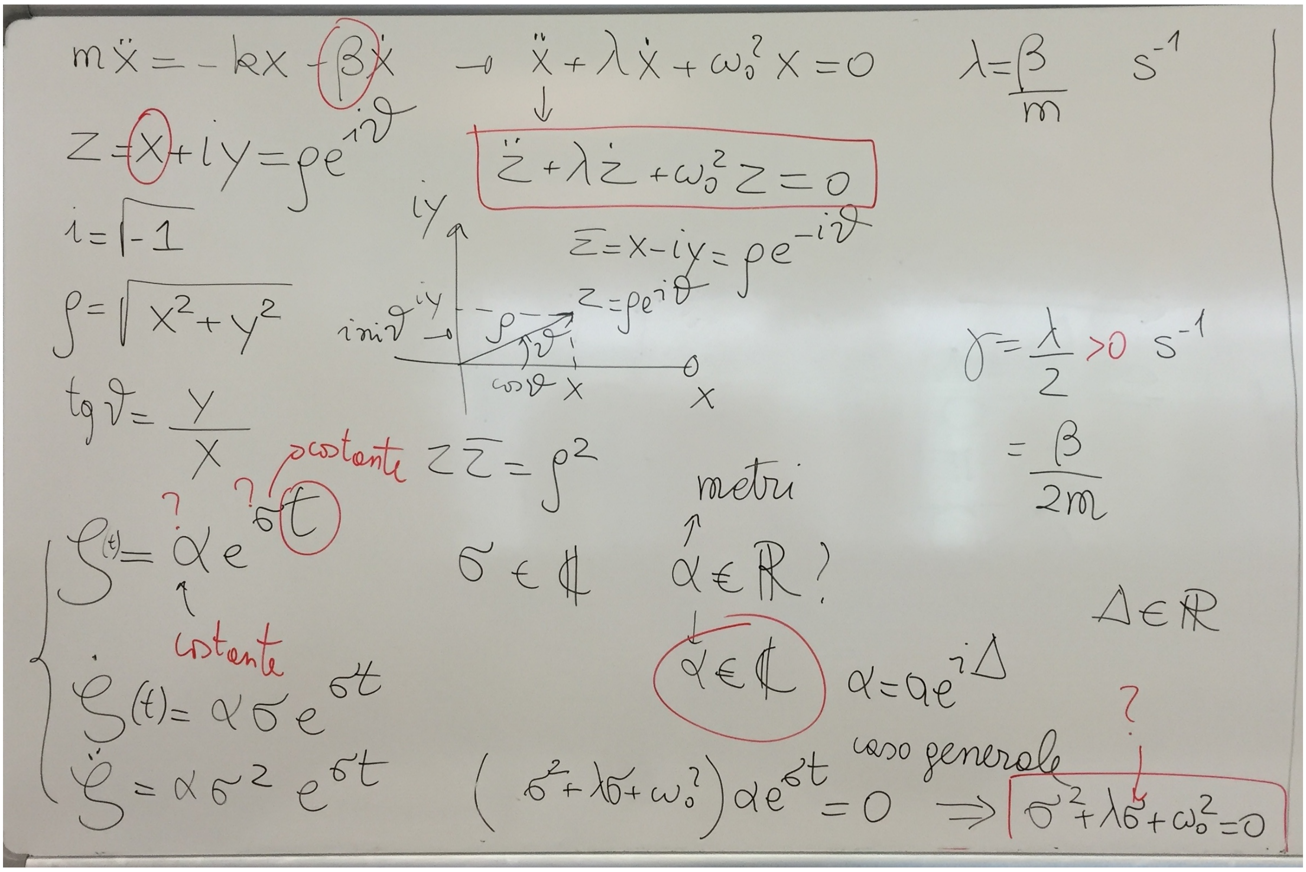
S7 - Lezioni 12 e 14 Aprile 2016:

- Continuazione oscillatore armonico smorzato (in una dimensione). Equazione del moto e soluzione mediante passaggio alla variabile complessa
- Oscillatore armonico forzato e smorzato (con dissipazione). Equazione del moto e soluzione mediante passaggio alla variabile complessa
- Concetto di risonanza
- Definizione e rilevanza del fattore di qualità di un oscillatore



$$\sigma_\pm = -\frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} - \omega_0^2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

1. $\omega_0 > \gamma \Rightarrow \sigma_\pm = -\gamma \pm \sqrt{-\underbrace{(\omega_0^2 - \gamma^2)}_{>0}} = -\gamma \pm \overbrace{\sqrt{-1}}^{i}\underbrace{\sqrt{\omega_0^2 - \gamma^2}}_{\omega < \omega_0}$

OSCILLATORE SMORZATO

$= -\gamma \pm i\omega$

$$\varsigma_\pm(t) = a e^{i\Delta} e^{(-\gamma \pm i\omega)t} = a e^{-\gamma t} e^{i(\omega t + \Delta)}$$

2. $\omega_0 < \gamma \Rightarrow \sigma_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$, $\quad \sqrt{\gamma^2 - \omega_0^2} < \gamma$

$$\varsigma_\pm(t) = a e^{i\Delta} e^{(-\gamma \pm \sqrt{\gamma^2 - \omega_0^2})t}$$

$-$: $a e^{i\Delta} e^{-(\gamma + \underbrace{\sqrt{\gamma^2 - \omega_0^2}}_{>0})t}$ $\qquad$ $+$: $a e^{i\Delta} e^{-\overbrace{(\gamma - \sqrt{\gamma^2 - \omega_0^2})}^{>0}t}$

OSCILLATORE SOVRASMORZATO

SMORZAMENTO CRITICO

3. $\omega_0 = \gamma$

$$\varsigma(t) = a e^{i\Delta} e^{-\gamma t}$$

OSCILLATORE ARMONICO FORZATO & SMORZATO

$$m\ddot{x} = -kx - \beta\dot{x} + F(t) \to \boxed{\ddot{x} + \lambda\dot{x} + \omega_0^2 x = \frac{F(t)}{m}}$$

$$\ddot{z} + \lambda\dot{z} + \omega_0^2 z = \frac{\mathcal{F}(t)}{m}$$

$\mathcal{F} = F_0 e^{i\delta} e^{i\omega t} = F_0 e^{i(\omega t + \delta)}$ $\qquad$ $F_0 \in \mathbb{R}$, $\omega \in \mathbb{R}$, $\delta \in \mathbb{R}$

$\varsigma(t) = \alpha e^{i\omega t}$ $\qquad$ $\alpha = a e^{i\Delta}$

$\dot{\varsigma}(t) = i\omega\alpha e^{i\omega t}$; $\quad \ddot{\varsigma} = -\omega^2 \alpha e^{i\omega t}$

$$-\omega^2 \alpha e^{i\omega t} + i\lambda\omega\alpha e^{i\omega t} + \omega_0^2 \alpha e^{i\omega t} = \frac{F_0}{m} e^{i\delta} e^{i\omega t}$$

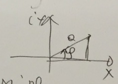
$$(-\omega^2 + i\lambda\omega + \omega_0^2)\,\alpha \underbrace{e^{i\omega t} e^{-i\omega t}}_{1} = \frac{F_0}{m} e^{i\delta} \underbrace{e^{i\omega t} \cdot e^{-i\omega t}}_{1}$$

$$(\omega_0^2 - \omega^2 + i\lambda\omega)\, a e^{i\Delta} = \frac{F_0}{m} e^{i\delta}$$

$$a e^{i\Delta} = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\lambda\omega} e^{i\delta} \Rightarrow \varsigma(t) = \underbrace{\frac{F_0/m}{\omega_0^2 - \omega^2 + i\lambda\omega}}_{R} \underbrace{e^{i\delta} e^{i\omega t}}_{e^{i(\omega t + \delta)}}$$

$$R = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\lambda\omega} \equiv \rho e^{i\vartheta}$$

$$\frac{1}{R} = \frac{\omega_0^2 - \omega^2 + i\omega\lambda}{F_0/m} = \frac{1}{\rho} e^{-i\vartheta}$$



$$\frac{1}{\rho} e^{-i\vartheta} = \left(\frac{\omega_0^2 - \omega^2}{F_0/m}\right) + i\left(\frac{\omega\lambda}{F_0/m}\right)$$

$x + iy$

$\mathrm{tg}\,\varphi = \frac{\sin\varphi}{\cos\varphi}$ $\qquad$ $a e^{i\varphi} = a\cos\varphi + ia\sin\varphi$

$a^2 = a^2\cos^2\varphi + a^2\sin^2\varphi = a^2$

$$\left(\frac{1}{\rho}\right)^2 = \frac{(\omega_0^2 - \omega^2)^2 + \omega^2\lambda^2}{F_0^2/m^2} \qquad \mathrm{tg}(-\vartheta) = \frac{\omega\lambda}{\omega_0^2 - \omega^2} = -\mathrm{tg}\,\vartheta$$

$$\rho = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\lambda^2}}$$

$$\varsigma(t) = \rho e^{i\vartheta} e^{i(\omega t + \delta)} =$$

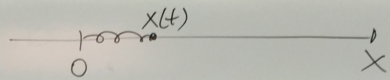
$$\boxed{\varsigma(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\lambda^2}} e^{i(\omega t + \delta + \vartheta)}}$$

OSCILLATORE ARMONICO FORZATO CON DISSIPAZIONE

$$F(t) = F_0 \cos(\omega t + \delta) \qquad F_0 \in \mathbb{R}$$

$$\mathcal{F}(t) = F_0 e^{i(\omega t + \delta)} \qquad \delta \in \mathbb{R}$$

$$m\ddot{x} = -kx - \beta\dot{x} + F(t)$$



$$\boxed{\ddot{x} + \lambda\dot{x} + \omega_0^2 x = \frac{F(t)}{m}}$$

$$\lambda = \frac{\beta}{m} \quad s^{-1} \qquad \omega_0^2 = \frac{k}{m}$$

$$z = x + iy$$

$$\boxed{\ddot{z} + \lambda\dot{z} + \omega_0^2 z = \frac{\mathcal{F}(t)}{m}}$$

$$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$$

? $$\xi(t) = \alpha e^{i\omega t} \qquad \alpha \in \mathbb{C}$$

$$\alpha = a e^{i\Delta} \qquad a, \Delta \in \mathbb{R} \quad m$$

$$\dot{\xi} = i\omega\alpha e^{i\omega t} \qquad \ddot{\xi} = -\omega^2\alpha e^{i\omega t}$$

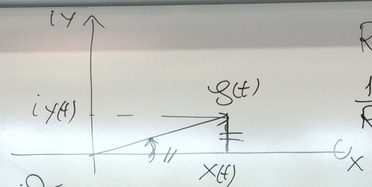
$$\times e^{-i\omega t}$$

$$-\omega^2\alpha \cancel{e^{i\omega t}} + i\omega\lambda\alpha \cancel{e^{i\omega t}} + \omega_0^2\alpha \cancel{e^{i\omega t}} = \frac{F_0}{m} e^{i\delta} \cancel{e^{i\omega t}} \qquad m\,\cancel{s^{-2}}$$

$$(\omega_0^2 - \omega^2 + i\omega\lambda) a e^{i\Delta} = \frac{F_0}{m} e^{i\delta} \Longrightarrow \underbrace{a e^{i\Delta}}_{m} = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\omega\lambda} e^{i\delta} \quad s^{-2}$$

$$\xi(t) = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\omega\lambda} e^{i(\omega t + \delta)}$$

$$\xi(t) = R e^{i(\omega t + \delta)} \qquad R \in \mathbb{C} \qquad R = \rho e^{i\vartheta}$$



$$\rho \in \mathbb{R}, \ \vartheta \in \mathbb{R}$$

$$= \rho e^{i(\omega t + \delta + \vartheta)}$$

$$= \rho\left(\cos(\omega t + \delta + \vartheta) + i\sin(\omega t + \delta + \vartheta)\right)$$

$$R = \rho e^{i\vartheta} \qquad \rho^2 = a^2 + b^2$$

$$\frac{1}{R} = \frac{1}{\rho e^{i\vartheta}} = \frac{1}{\rho} e^{-i\vartheta} = a + ib$$

$$= \frac{1}{\rho}(\cos\vartheta + i\sin\vartheta)$$

$$\mathrm{tg}(-\vartheta) = \frac{b}{a}$$

$$= -\mathrm{tg}\,\vartheta$$

$$E \propto A^2$$

$$\downarrow$$

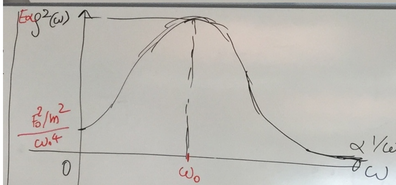
$$\frac{1}{R} = \frac{\omega_0^2 - \omega^2 + i\omega\lambda}{F_0/m} = \underbrace{\left(\frac{\omega_0^2 - \omega^2}{F_0/m}\right)}_{=a} + i\underbrace{\left(\frac{\omega\lambda}{F_0/m}\right)}_{=b}$$

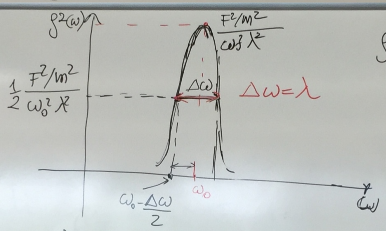
$$\rho^2(\omega) = \frac{F_0^2/m^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\lambda^2}$$

$$\frac{1}{\rho^2} = \frac{(\omega_0^2 - \omega^2)^2 + \omega^2\lambda^2}{F_0^2/m^2}$$

$$\mathrm{tg}(-\vartheta) = \frac{\omega\lambda}{\omega_0^2 - \omega^2}$$

$$= -\mathrm{tg}\,\vartheta$$





$$\rho^2\left(\omega_0 - \frac{\Delta\omega}{2}\right) = \frac{\cancel{F_0^2/m^2}}{4\omega_0^2\left(\cancel{\omega_0} - \cancel{\omega_0} + \frac{\Delta\omega}{2}\right)^2 + \omega_0^2\lambda^2} = \frac{1}{2}\frac{\cancel{F^2/m^2}}{\omega_0^2\lambda^2}$$

$$\cancel{4}\omega_0^2\frac{\Delta\omega^2}{\cancel{4}} + \omega_0^2\lambda^2 = 2\omega_0^2\lambda^2$$

$$\cancel{\omega_0^2}\Delta\omega^2 = \cancel{\omega_0^2}\lambda^2$$

$$\boxed{\Delta\omega = \lambda}$$

$$\rho^2(0) = \frac{F_0^2/m^2}{\omega_0^4}$$

$$\omega \simeq \omega_0$$

$$\rho^2(\omega = \omega_0) = \frac{F_0^2/m^2}{\omega_0^2\lambda^2}$$

RISONANZA

$$\omega \to +\infty$$

$$\rho^2 = \frac{F^2/m^2}{[(\omega_0 + \omega)(\omega_0 - \omega)]^2 + \omega^2\lambda^2} \simeq \frac{F^2/m^2}{4\omega_0^2(\omega_0 - \omega)^2 + \omega_0^2\lambda^2}$$

$$\rho^2(\omega = \omega_0) \simeq \frac{F^2/m^2}{\omega_0^2\lambda^2}$$

$$\sqrt{k/m}$$

$$Q = 2\pi\,\frac{\text{Energia immagazzinata in 1 ciclo } P_0 = 2\pi/\omega_0}{\text{Energia dissipata in un periodo } P_0}$$

$$Q = 2\pi\frac{E}{(\Delta E)_{P_0}}$$

$$\Delta t \qquad \boxed{(\Delta E)_{\Delta t} = -\frac{2\pi}{Q} E \frac{\Delta t}{P_0}}$$

$$\Delta t \to \underset{\substack{\uparrow \\ dE}}{dt}$$

$$\boxed{dE = -\frac{2\pi}{Q} E \frac{dt}{P_0}}$$

$$\frac{dE}{E} = -\frac{1}{Q}\underbrace{\frac{2\pi}{P_0}}_{=\omega_0} dt$$

$$\boxed{\int_{E_0}^{E(t)} \frac{dE'}{E'} = -\frac{\omega_0}{Q}\int_{0}^{t} dt'}$$

$$\left[\ln E'\right]_{E_0}^{E} = -\frac{\omega_0}{Q} t$$

$$\ln E - \ln E_0 = -\frac{\omega_0}{Q} t$$

$$\ln \frac{E(t)}{E_0} = -\frac{\omega_0}{Q} t$$

$e \qquad e$

$$\frac{E(t)}{E_0} = e^{-\frac{\omega_0}{Q}t}$$

$$\boxed{E(t) = E_0 e^{-\frac{\omega_0}{Q}t}} = E_0 e^{-t/\tau}$$

$$\tau = \frac{Q}{\omega_0}$$

$$A \propto \sqrt{E}$$

$$A(t) = A_0 e^{-\frac{\omega_0}{2Q}t}$$

$$\frac{\omega_0}{\cancel{2}Q} = \frac{\lambda}{\cancel{2}}$$

$$Q = \frac{\omega_0 \to s^{-1}}{\lambda \to s^{-1}}$$