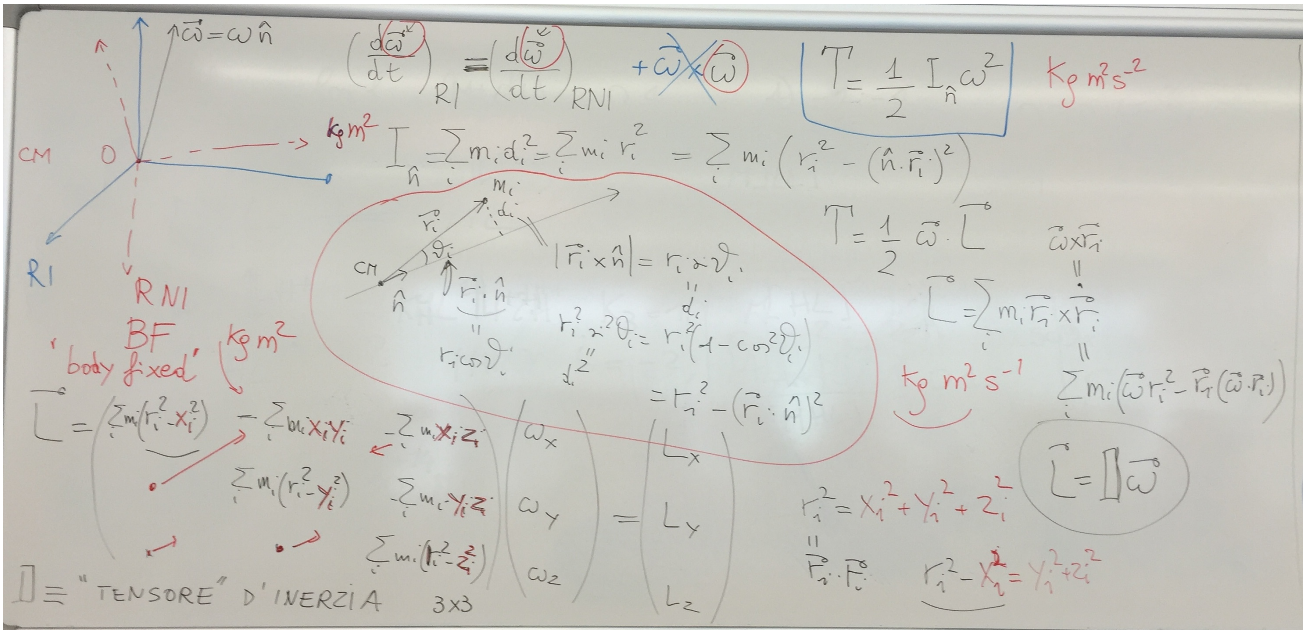
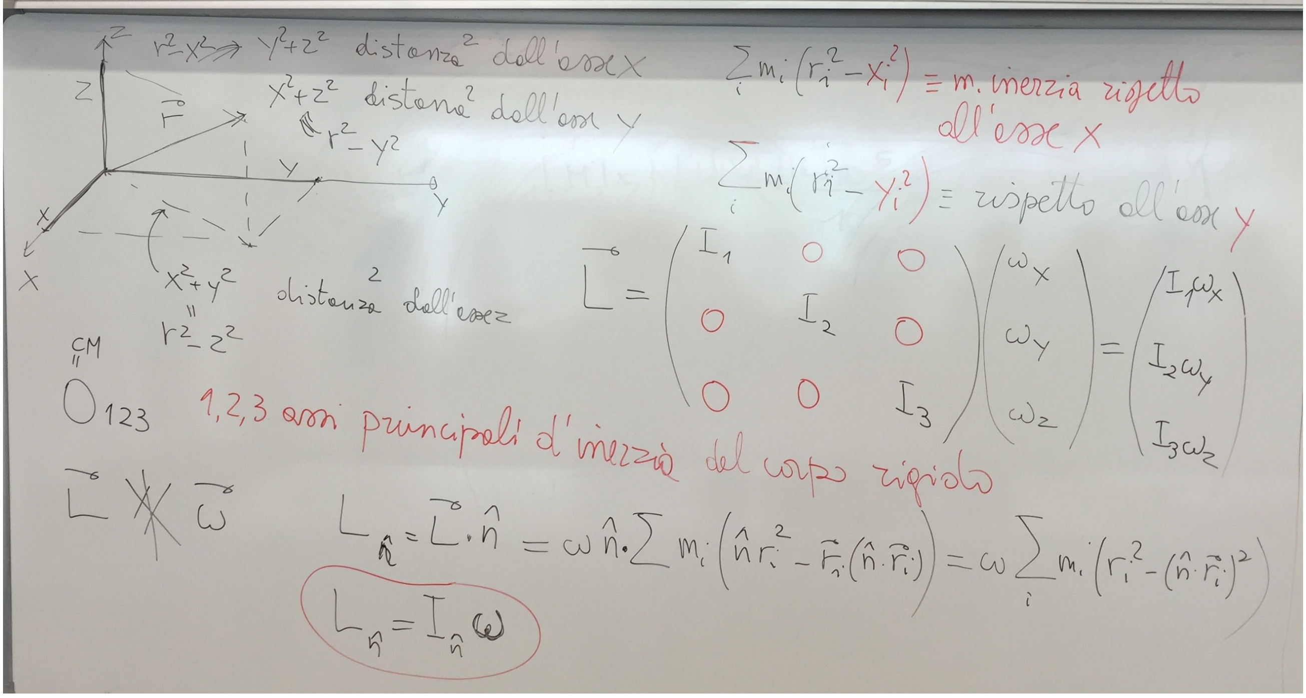
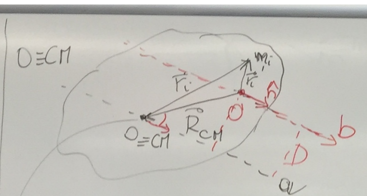
S9 - Lezioni 26 e 28 Aprile 2016:

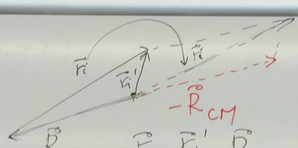
- Continuazione corpo rigido. Energia cinetica, momento angolare, momento di inerzia rispetto ad un asse
- Momento angolare, tensore d'inerzia, assi principali di inerzia, momenti principali di inerzia
- Calcolo dei momenti di inerzia in vari casi





O ≡ CM





$\vec{r}_i = \vec{r}_i' - \vec{R}_{CM}$

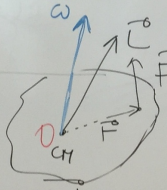
$M = \sum_i m_i \qquad \vec{R}_{CM} = \frac{\sum_i m_i \vec{r}_i'}{M}$

a//b

$$I_b = \sum_i m_i (\vec{r}_i' \times \hat{n})^2 = \sum_i m_i \left((\vec{r}_i + \vec{R}_{CM}) \times \hat{n}\right)^2 = \sum_i m_i (\vec{r}_i \times \hat{n})^2 + M(\vec{R}_{CM} \times \hat{n})^2 \overset{?}{+} \cancel{2(\vec{R}_{CM} \times \hat{n}) \cdot \left(\sum_i m_i \vec{r}_i \times \hat{n}\right)}$$

$$\to \vec{O} = \frac{\sum_i m_i \vec{r}_i}{M}$$

$$= M(\vec{R}_{CM} \times \hat{n})^2 + \underbrace{\sum_i m_i (\vec{r}_i \times \hat{n})^2}_{I_a} = I_a + M\left(\text{distanza tra a e b minima } D\right)^2$$



$$\vec{N} = \vec{r} \times \vec{F}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{RI} = \vec{N}$$

Se x,y,z : assi principali di inerzia

$$\vec{L} = (I_x \omega_x, I_y \omega_y, I_z \omega_z) \qquad (\dot{\vec{L}})_{RNI \equiv BF\,xyz} = (I_x \dot{\omega}_x, I_y \dot{\omega}_y, I_z \dot{\omega}_z)$$

$$\left(\frac{d'\vec{L}}{dt}\right)_{RNI \leftarrow BF} + \vec{\omega} \times \vec{L} = \vec{N}$$

$$N_x = I_x \dot{\omega}_x + (\omega_y L_z - \omega_z L_y) = \boxed{I_x \dot{\omega}_x - \omega_y \omega_z (I_y - I_z) = N_x}$$
$$N_y = I_y \dot{\omega}_y + (\omega_z L_x - \omega_x L_z) = \boxed{I_y \dot{\omega}_y - \omega_z \omega_x (I_z - I_x) = N_y}$$
$$N_z = I_z \dot{\omega}_z + (\omega_x L_y - \omega_y L_x) = \boxed{I_z \dot{\omega}_z - \omega_x \omega_y (I_x - I_z) = N_z}$$

← Eq. di Eulero per il moto del corpo rigido rispetto al suo centro di massa (O)

Oxyz sistema BF degli assi principali d'inerzia

↑ I_x, I_y, I_z

$\vec{\omega}$, $\vec{N}$

CALCOLO DEI MOMENTI D'INERZIA

$$N_x = I_x \dot{\omega}_x + (\omega_y L_z - \omega_z L_y) = \boxed{I_x \dot{\omega}_x - \omega_y \omega_z (I_y - I_z) = N_x}$$
$$N_y = I_y \dot{\omega}_y + (\omega_z L_x - \omega_x L_z) = \boxed{I_y \dot{\omega}_y - \omega_z \omega_x (I_z - I_x) = N_y}$$
$$N_z = I_z \dot{\omega}_z + (\omega_x L_y - \omega_y L_x) = \boxed{I_z \dot{\omega}_z - \omega_x \omega_y (I_x - I_z) = N_z}$$

← Eq. di Eulero per il moto del corpo rigido rispetto al suo centro di massa (O)

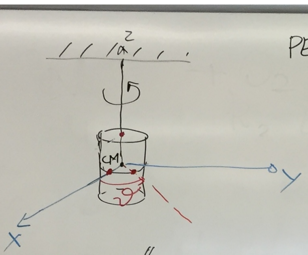
Oxyz sistema BF degli assi principali d'inerzia

↑ I_x, I_y, I_z

$\vec{\omega}$, $\vec{N}$

CALCOLO DEI MOMENTI D'INERZIA

$$\left(\frac{d\vec{L}}{dt}\right)_{RI} = \vec{N}$$



PENDOLO DI TORSIONE

$$N = -k_{tor}\,\vartheta(t)$$

$$\mathbb{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \cancel{\omega_x} \\ \cancel{\omega_y} \\ \omega_z \end{pmatrix} \qquad \vec{L} = \mathbb{I}\vec{\omega} \qquad \dot{\vartheta}$$

$$L_z = I_3\,\omega_z$$

I_3 rispetto z

scelto assi principali di inerzia del cilindro

$$[k_{tor}] = N_{ewton} \cdot m$$

$$\vec{N} = (0,\ 0,\ N)$$

$$\vec{L} = (0,\ 0,\ L) \qquad L = I_3\dot{\vartheta}$$

$$(F = -kx)$$

$$[k] = \frac{Newton}{m}$$

$$\vec{L} = \begin{pmatrix} 0 \\ 0 \\ I_3\omega_z \end{pmatrix} \qquad \omega_z = \dot{\vartheta}$$

$$\omega_0^2 = \frac{k_{tor}}{I_3} \qquad \frac{\cancel{kg}\,m^2 s^{-2}}{\cancel{kg}\,m^2}$$

$$\ddot{\vartheta} + \omega_0^2\vartheta = 0$$

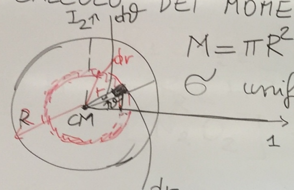
$$\left(\frac{dL}{dt}\right)_{RI} = -k_{tor}\vartheta \qquad \frac{dL}{dt} = I_3\ddot{\vartheta}$$

$$I_3\ddot{\vartheta}(t) = -k_{tor}\vartheta(t)$$

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I_3}{k_{tor}}}$$

$$\boxed{\ddot{\vartheta} + \frac{k_{tor}}{I_3}\vartheta = 0}$$

CALCOLO DEI MOMENTI D'INERZIA

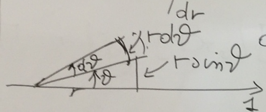


$$M = \pi R^2 \sigma$$

σ uniforme (omogeneo)

$$I_3 = \int_M \rho^2\, dm = \int_M r^2\, dm = 2\pi\sigma\int_0^R r^3\, dr = 2\pi\sigma\frac{R^4}{4} = \frac{1}{2} M R^2 \qquad \sigma = M/\pi R^2$$

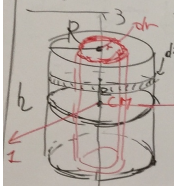
$$dm = \sigma\, 2\pi r\, dr$$



$$I_1 = I_2 = \int_M r^2\sin^2\vartheta\, dm = \sigma\int_0^{2\pi}\sin^2\vartheta\, d\vartheta\int_0^R r^3\, dr = \sigma\,\frac{1}{2}\,2\pi\,\frac{R^4}{4} = \frac{M}{\pi R^2}\pi\frac{R^4}{4} = \frac{1}{4} M R^2$$

$$dm = \sigma\, r\, dr\, d\vartheta$$

$$\int \sin^2\vartheta\, d\vartheta = \int \cos^2\vartheta\, d\vartheta = \frac{1}{2}\int(\sin^2\vartheta + \cos^2\vartheta)\, d\vartheta = \frac{1}{2}\int d\vartheta$$



$$M = \rho\pi R^2 h$$

ρ $kg\, m^{-3}$

$$I_3 = \int_M r^2\, dm = 2\pi\rho h\int_0^R r^3\, dr = 2\pi h\rho\frac{R^4}{4} = 2\pi h\frac{M}{\pi R^2 h}\frac{R^4}{4} = \frac{1}{2} M R^2$$

non c'è h

$$dm = \rho\, 2\pi r h\, dr$$

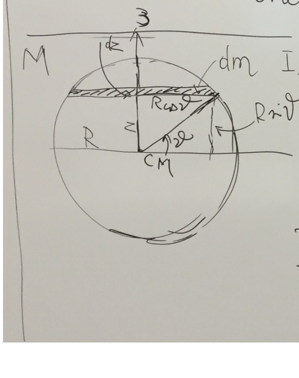
$$I_1 = I_2 = \int_M \frac{1}{4}R^2\, dm + \int_M z^2\, dm = \frac{1}{4} M R^2 + 2\pi\rho R^2\int_0^{h/2} z^2\, dz = \frac{1}{4} M R^2 + 2\pi\rho R^2\frac{h^3}{8\cdot 3} = \frac{1}{4} M R^2 + \frac{1}{12} M h^2$$

$$\int_M \frac{1}{4}R^2\, dm = \frac{1}{4} M R^2 \qquad \rho = \frac{M}{\pi R^2 h}$$

$$dm = \rho\pi R^2\, dz$$

massa del cilindro

massa del dischetto sottile



$$I_1 = I_2 = I_3 \qquad \rho \qquad M = \rho\frac{4}{3}\pi R^3$$

$$dm = \rho\pi R^2\cos^2\vartheta\, dz = \rho\pi(R^2 - z^2)\, dz$$

$$R^2\cos^2\vartheta + R^2\sin^2\vartheta = R^2 \qquad R^2\cos^2\vartheta = R^2 - z^2$$

$$dI = \frac{1}{2}R^2\cos^2\vartheta\, dm = \frac{1}{2}\pi\rho(R^2 - z^2)^2\, dz$$

$$I_3 = 2\cdot\frac{1}{2}\pi\rho\int_0^R (R^2 - z^2)^2\, dz = \frac{2}{5} M R^2$$