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Can the equivalence principle be tested with freely orbiting masses?

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Abstract

The orbit of a free-flying test mass having differing inertial and gravitational masses relative to a reference mass obeying the equivalence principle (EP), depends on the EP-violation e as well as on the release conditions. For every orbit a release error does exist which compensates for the EP effect. The secular term in the relative distance of the masses (due to orbit period changes) is also dependent on a combination of e and release errors in a non-separable way.

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1. Circular orbits with identical periods

Let us first determine the difference in radius of two circular orbits having the same orbital frequency when the masses differ in the ratio of gravitational to inertial mass. We define this ratio by

$$m_g = m_i(1+e). \tag{1}$$

Figure 1 shows this situation and defines the coordinate system used throughout this paper. This reference system (y, z) is attached to a mass assumed to obey the equivalence principle (EP) on a circular orbit of radius r_0 (e.g. a heavy spacecraft). The system (y, z) is co-rotating. We then have for the orbital angular frequency

$$\omega_0 = \sqrt{M_E G/r_0^3}. \tag{2}$$

The difference in orbital radius $\mathrm{d}z$ is obtained from the equation

$$m_i(r_0 + \mathrm{d}z)\omega^2 = m_g \frac{GM}{(r_0 + \mathrm{d}z)^2} \tag{3}$$

which leads to

$$\mathrm{d}z = er_0/3. \tag{4}$$

This already shows the problem: if the mass with an EP-violation e is released at rest in the system (y, z) at the coordinate $z = \mathrm{d}z$, the orbital frequency will be the same, meaning that it will stay at rest.

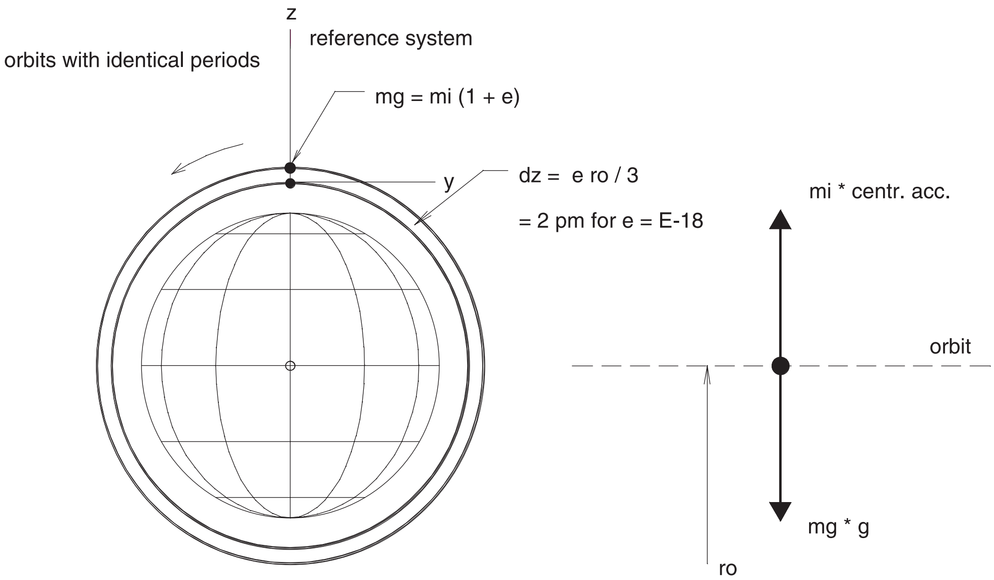

Figure 1. Coordinate system for comparing orbits of masses with differing ratios of inertial to gravitational mass.

2. Changes in orbital frequency

Let us now determine the secular term in the distance of the freely orbiting mass relative to the origin of the (y, z) system. This term is due to changes in the orbital frequency with the coordinate z_1 of the release point as well as in the function of the EP-violation e.

To avoid calculating the actual Kepler orbit we can use the conservation laws of total energy as well as of angular momentum between the two apsides (perigee and apogee). We have the two equations

$$\begin{aligned} &v_1(r_0 + z_1) = v_2(r_0 + z_2) \\ &v_1^2 - 2MG(1+e)/(r_0 + z_1) = v_2^2 - 2MG(1+e)/(r_0 + z_2). \end{aligned} \tag{5}$$

We first solve these for a mass with $e = 0$ released at rest in the reference system at $z = z_1$ (and $y = 0$) and then for a mass with e released at rest at the origin. As the equations involved are of higher degree, the solutions contain many terms. Considering r_0 as very large with respect to the coordinates and e as very small compared with unity and restricting ourselves to linear terms in the coordinates, we obtain for z_2, which is the z-coordinate of the mass at the opposite apside

$$\begin{aligned} &\text{with} \quad e = 0 \qquad\quad z_2 = 7z_1 \\ &\text{with} \quad z_1 = 0 \qquad\quad z_2 = -2er_0. \end{aligned} \tag{6}$$

The semi-major axis a of the orbit is then

$$a = (2r_0 + z_1 + z_2)/2. \tag{7}$$

We can now use Kepler's law to find the change in orbital frequency

$$2\,\mathrm{d}\omega/\omega = -3\,\mathrm{d}a/a. \tag{8}$$

Combining both cases we finally obtain

$$\mathrm{d}\omega/\omega = -6z_1/r_0 + 3e/2. \tag{9}$$

This confirms that there is indeed no way of distinguishing whether an observed orbital frequency change is due to an EP-violation e or to a release error z_1. The value giving no change $\mathrm{d}\omega$ is $z_1 = er_0/4$. This is not exactly $\mathrm{d}z$ of equation (4), as the orbits are different in the two cases.

Jafry [1], analysing a particular proposal for an EP-experiment, arrived at essentially the same conclusion.

3. Modifying Hill's equations and examples of orbits

The equations of motion for the test mass relative to the reference system are Hill's equations. For motion in the (y, z) orbit plane without any external forces they are (with $\omega = \omega_0$)

$$\begin{aligned}\ddot{y} &= 2\omega\dot{z}\\ \ddot{z} &= -2\omega\dot{y} + 3\omega^2 z.\end{aligned} \tag{10}$$

If we now have a test mass violating the EP by e, we have to modify these equations. The first terms are Coriolis accelerations and the last one is a combination of the gravity gradient and the centrifugal acceleration. Indeed, in the term $3\omega^2 z$ we have two terms:

- the first (a factor of 2) is due to the gravity gradient, and is therefore proportional to the gravitational mass;
- the second (a factor of 1) is the centrifugal acceleration in the co-rotating system, and is therefore proportional to the inertial mass (the same term is, in fact, removing the gravity gradient term normally present in the equation for y).

To derive the correct equations, including an EP-violation e, we take as the gravitation potential

$$MG(1+e)/r, \tag{11}$$

expand the gradient relative to the origin of the reference system, and add the centrifugal accelerations due to its co-rotation, taking care to use the inertial and gravitational masses where they belong. Then we subtract the inertial acceleration of the origin of the reference system (obeying the EP).

If, among the many terms obtained, we again take only those linear in the coordinates and drop terms containing coordinates divided by r_0, we obtain the new Hill's equations applicable to an EP-violating mass

$$\begin{aligned}\ddot{y} &= 2\omega\dot{z}\\ \ddot{z} &= -2\omega\dot{y} + \omega^2\{z(3+2e) - er_0\}.\end{aligned} \tag{12}$$

We can now easily look at different types of relative orbits obtained for various values of e and for release conditions including offsets and also initial velocities as they would be produced by real release mechanisms. (In the integration programme we also neglect $2e$ relative to 3.) Figure 2 shows the relative motion during one orbital period.

Figure 3 shows that the same rule holds even for the complicated orbits obtained with offsets in any direction combined with arbitrary initial release velocities. This shows that indeed an offset $\mathrm{d}z$ exactly cancels the effect of an EP-violation. This is obvious from equation (12) where we can write the last term (with $e \ll 1$) as $3\omega^2(z - \mathrm{d}z)$. The deeper reason for this is

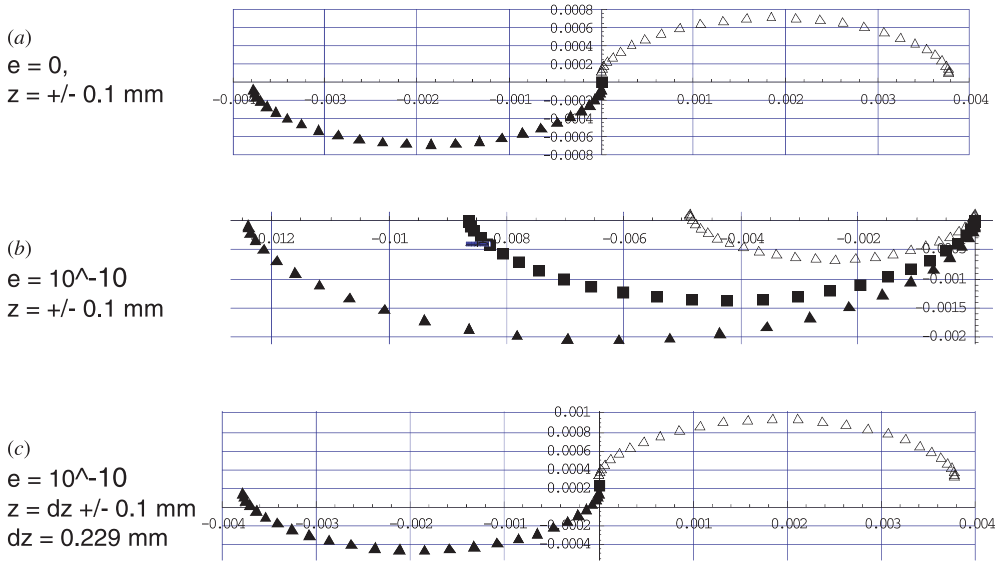

Figure 2. Motion of a test mass in the orbiting and co-rotating reference frame. The EP-violation e and the release offsets z used in the three cases are indicated. Cases (a) and (c) show that the effects of e and a and the corresponding offset of dz (see text) cannot be distinguished.

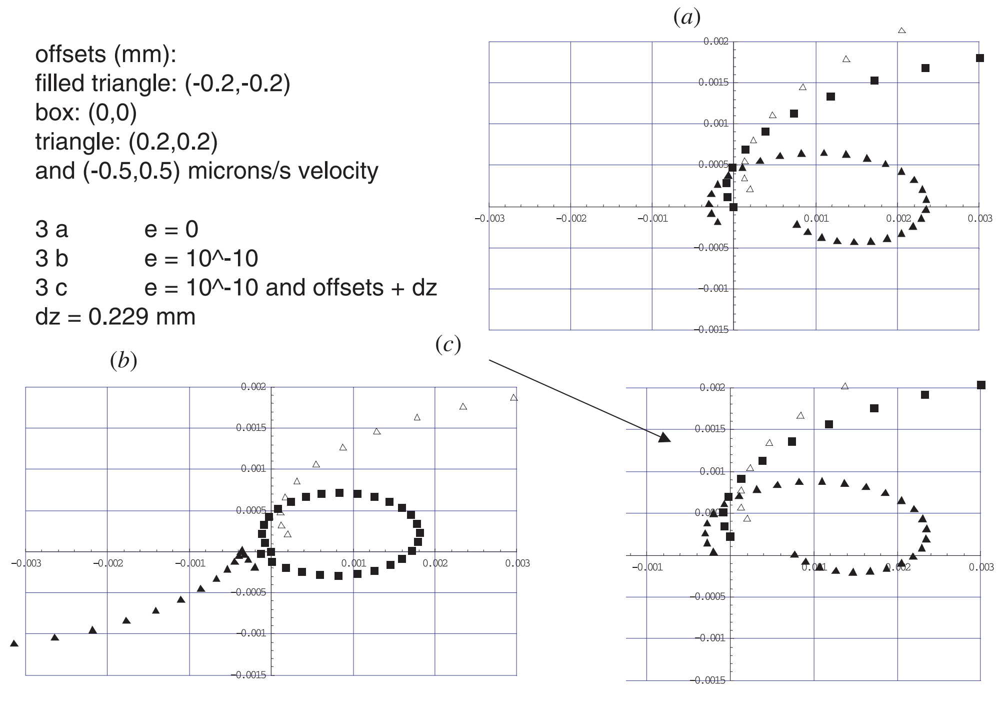

Figure 3. As figure 2, but the mass is released with both a non-radial offset and an initial velocity. A radial shift by dz still exactly cancels the effect of e.

that, to first order, increasing gravity by the factor $(1+e)$ just results in shifting the tidal field pattern outward by dz.

As an instructive test we can try to explain the value of the preceding motion of d$y = 3.8$ mm per orbit for the mass released at $z = -0.1$ mm in figure 2(a) (full triangles). From equation (9) we can determine the change dω in orbital frequency produced by the offset. Then there is a change in orbital velocity d$v = r_0\,\mathrm{d}\omega$ leading after the time T (orbit period) to a displacement dy,

$$\begin{aligned} \mathrm{d}\omega &= -6\omega z_1/r_0 \\ \mathrm{d}y &= \mathrm{d}v\,T = r_0\,\mathrm{d}\omega\,2\pi/\omega = -12\pi z_1 = 3.8\ \text{mm}. \end{aligned} \tag{13}$$

4. Conclusions

If we consider measuring the distance between two freely orbiting masses each having a particular EP-violation e, we have both periodic and secular terms. Both these terms, according to celestial mechanics, are determined by a combination of e and the release conditions (position and velocity) that cannot be separated.

Although in principle the release conditions can be controlled or measured, the precision achievable in practice is absolutely insufficient to measure an EP-violation. Indeed, assuming that a release error, which has to refer to the centres of mass (CM), can hardly be better than, say, 10 μm, this means, according to equation (9), that an EP-violation of 6×10^{-12} cannot even be detected. For the 10^{-18} aimed at by STEP, the release accuracy for the centre of mass would have to be 1.7 pm, a small fraction of an atomic diameter.

Therefore, the only way to determine the effect of the EP-violation is to consider the two equilibrium orbits of section 1. These being unstable, the masses necessarily have to be confined (and damped). Two different ways to do this have been proposed.

(a) In the clever concept of GG, the CM of the two masses confined by super-critical rotation actually exactly follow the orbits of figure 1. Measuring the displacement from outside is, however, critical, as an EP signal has the same frequency as the many disturbances expected to occur at the large rotation frequency.
(b) In STEP, a differential accelerometer is rotated in the plane of orbit and thereby the test masses will try to align their CM to the corresponding orbits in figure 1. The motion of the masses bound by springs is, however, very complicated [2, 3], but the EP signal can be made to occur at a frequency incommensurable with both the frequency of orbit and that of rotation, where disturbances are expected.

Why then, does the *Lunar laser ranging* (LLR) beautifully used by Nordtvedt to search for EP violations in the Sun–Earth–Moon system work? Because the cases are different. In an experiment with masses freely orbiting the Earth, we have seen that the two orbits are fixed in space but that their orbit elements depend on e and the initial position and velocity at release in a non-separable way. The orbits are fixed in space only if one neglects the perturbations by the Earth gravity multipoles. To first order they act in the same way on both masses and in any case make the problem worse.

In LLR, the Earth–Moon binary system is considered. The heavy masses make it insensitive to any disturbances other than celestial perturbations. The perturbations of the Earth–Moon system by the Sun will be different if the Eötvös ratios e between Earth–Sun and Earth–Moon are not the same. That is the Nordtvedt effect.

Its equivalent in our case of Earth-orbiting test masses would be to determine the orbit perturbations of the Earth–test mass binary system by the Sun (or the Moon). As light masses suffer many disturbances, such as air drag or magnetic forces, radiation pressure, as well as very large perturbations by the oblateness and the higher-order gravity moments of the Earth, such a measurement is out of the question.

Along this line Nordtvedt has recently proposed to study the effect of EP violations on the action of the different materials of the Earth's crust on STEP. However, these effects are only detectable if a very large EP-violation, of the order of 10^{-14}, is present [4].

References

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