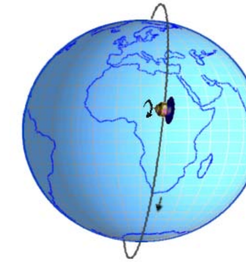


GGG

Un accelerometro differenziale in rotazione veloce per la verifica del principio di equivalenza

Anna Nobili e Raffaello Pegna

INFN Sezione di Pisa, 22 Giugno 2011

Why testing the Weak Equivalence Principle? Why in space?

- The Weak Equivalence Principle is the founding pillar of General Relativity

- EP tests are the only ones which test GR for composition dependence

- Evidence of EP violation would prove the existence of a new composition dependent force of Nature and make for a scientific revolution (by comparison, a geodetic or Lense Thirring precession or light rays deflection different from Einstein's prediction would simply call for another metric theory of gravity slightly different from GR…)

.. aim at higher sensitivity whenever the possibility for an improvement arises

<u>In low orbit around the Earth:</u>

- 3 orders of magnitude stronger signal
- weightlessness
- experiment isolated in space

State of the art

Authors	Apparatus	Source mass	Materials	$\eta \equiv \Delta a / a$
Eötvös et al. ≈1900 collected in Ann. Phys. **1922**	Torsion balance. Not rotating. No signal modulation	Earth	Many combinations	$10^{-8} \div 10^{-9}$
Roll, Krotkov & Dicke Ann. Phys. **1964**	Torsion balance. Not rotating. 24hr modulation by Earth rotation	Sun	Al – Au	$(1.3 \pm 1) \times 10^{-11}$
Braginsky & Panov JETP **1972**	Torsion balance. 8TMs. Not rotating. 24hr modulation by Earth rotation	Sun	Al – Pt	$(-0.3 \pm 0.9) \times \mathbf{10^{-12}}$
E. Fischbach et al.: "Reanalysis of the Eötvös Experiment" PRL 1986				
Eöt-Wash, PRD **1994**	<u>Rotating</u> torsion balance. ≈ 1hr modulation	Earth	Be – Cu	$(-1.9 \pm 2.5) \times \mathbf{10^{-12}}$
			Be – Al	$(-0.2 \pm 2.8) \times 10^{-12}$
Eöt-Wash, PRL 1999	<u>Rotating</u> torsion balance. 1hr to 36' modulation	Sun	Earthlike/ Moonlike	$\approx 10^{-12}$ (SEP 1.3×10^{-3})
Eöt-Wash, PRL **2008**	<u>Rotating</u> torsion balance. 20' modulation	Earth	Be – Ti	$(0.3 \pm 1.8) \times \mathbf{10^{-13}}$

36 yr

14 yr

GG: aiming at an improvement from 10^{-13} to 10^{-17}

- Slowly rotating torsion balances have achieved:

$$(\Delta a)_{TB} = 10^{-16} g \quad \Rightarrow \quad \eta_{TB} = 10^{-13}$$

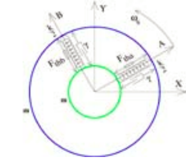
- In order to improve the EP test by 4 orders of magnitude GG must be able to sense differential accelerations between the test masses only one order of magnitude smaller than torsion balances, simply because in space the signal is 3 orders of magnitude stronger:

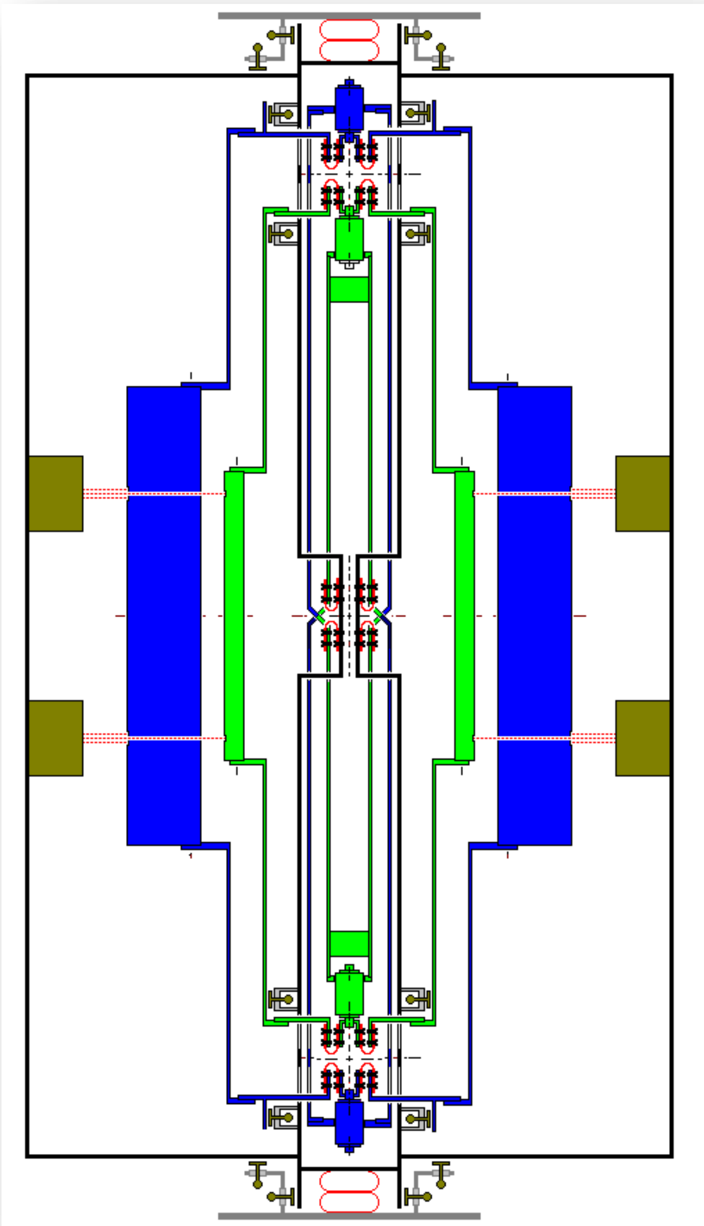
$$(\Delta a)_{GG} = 10^{-17} g \quad \Rightarrow \quad \eta_{GG} = 10^{-17}$$

Collaboration ongoing with JPL to submit GG to the next EXPLORER call of NASA as a NASA led small mission with ASI partnership (GG not submitted previous EXPLORER call last February because a key technology −FEEP thrusters− would not be tested on time by LISA/PF −more than 2 yrs launch delay of LISA/PF announced in January; NASA has withdrawn from LISA shortly afterwards)

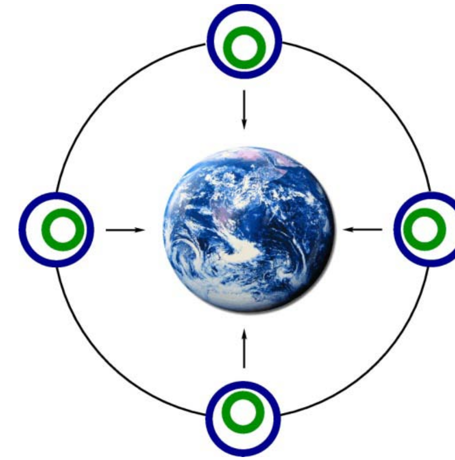
GG mission duration can be ensured by cold gas thrusters (baseline thrusters for GAIA).
For next NASA call:
– adjust mission design and error budget with new thrusters
– improve GGG prototype sensitivity at 1.7×10^{-4} Hz (frequency of EP violation signal in space)

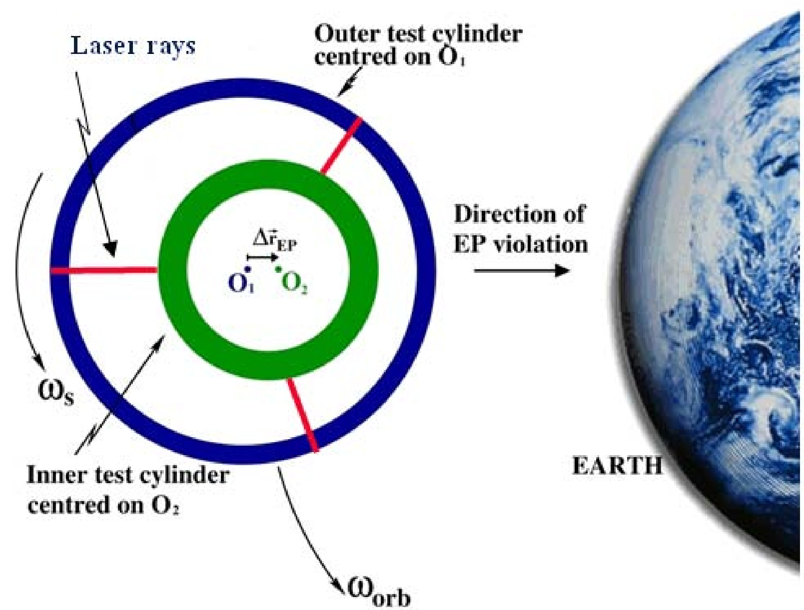
The GG experiment in space (I)





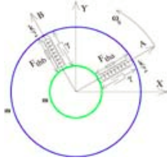
Put the concentric cylinders in LEO and spin around the symmetry axis so that the sensitive plane can detect differential accelerations acting in the orbit plane, e.g. an EP violation…

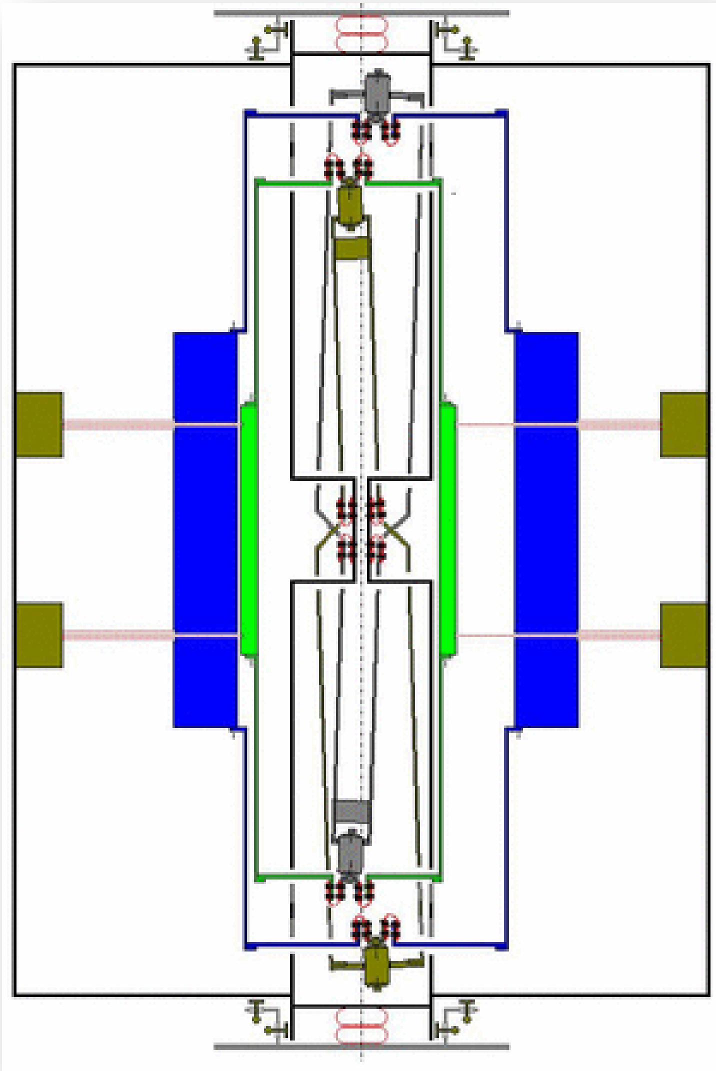




The GG experiment in space (II)

$\vec{\omega}_s$ $\quad \omega_s = 2\pi/1\,\mathrm{s}$





$\vec{n}$ $\quad n = 2\pi/5800\,\mathrm{s}$

$\vec{r}$

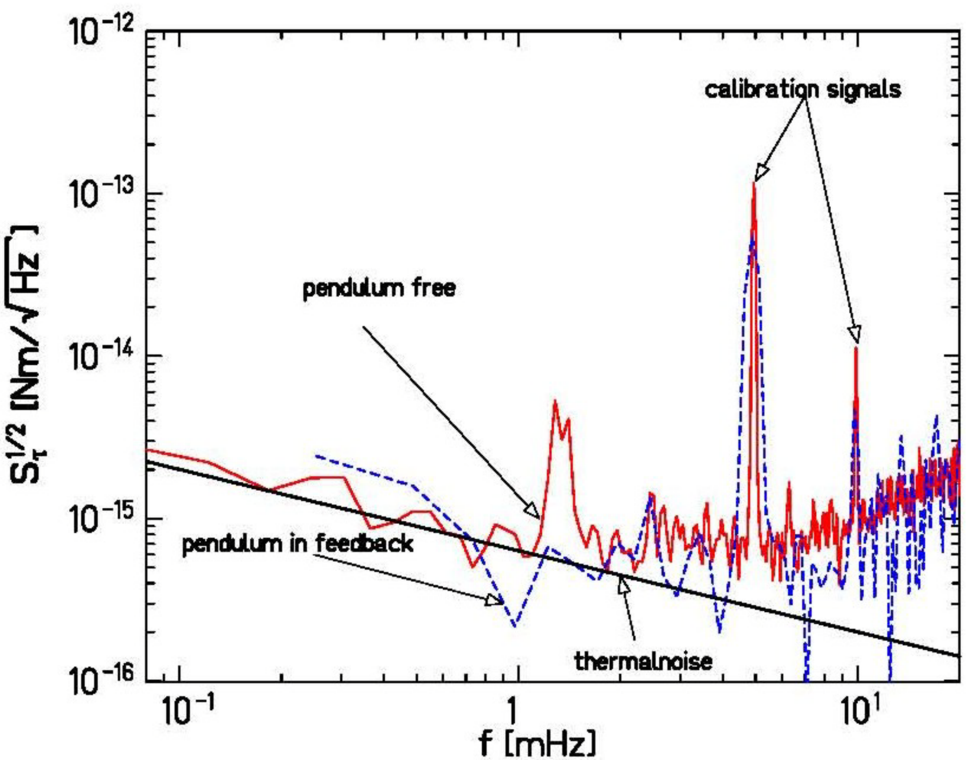
$r = R_\oplus + 600\,\mathrm{km}$

$$\eta = 10^{-17} \;, \; \omega_n = 2\pi/540\,\mathrm{s} \Longrightarrow \Delta r_{EP} \simeq 0.5\,\mathrm{pm}$$

Laser read-out (withJPL laser gauge tested for SIM)

Thermal noise & integration time

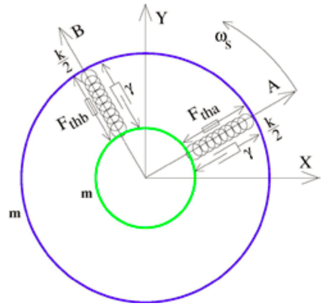
- Low thermal noise is crucial for a space experiment to ensure short integration time
- Modulation of the signal by rotation allows also reduction of thermal noise



Rotating torsion balances are operated at thermal noise level and demonstrate that it is reduced by rotation (Adelberget at al. 2009), but they are limited to rotation rates below the natural frequency, in order not to attenuate the expected signal

The GG experiment is unique in that it allows rotation rates well above the natural frequency (without signal attenuation…), hence a much stronger reduction of thermal noise, yielding a much shorter integration time

Abatement of thermal noise and integration time in GG



In the non rotating frame the force of the signal has low frequency:

$$(a_{th})|_{t_{int}} \simeq \sqrt{\frac{4KT\omega_n\phi(\omega_s)}{\mu}}\frac{1}{\sqrt{(\omega_s/\omega_n)}}\frac{1}{\sqrt{t_{int}}}$$

$$\omega_s \gg \omega_n$$

In GG (10^{-17} target in EP test, 10^{-17}g to be measured, 1Hz spin rate):

$$\omega_s/\omega_n \simeq 540 \quad \phi(\omega_s) \simeq 1/20000 \;\; m = 10\text{kg} \;\; T = 300\text{K}$$

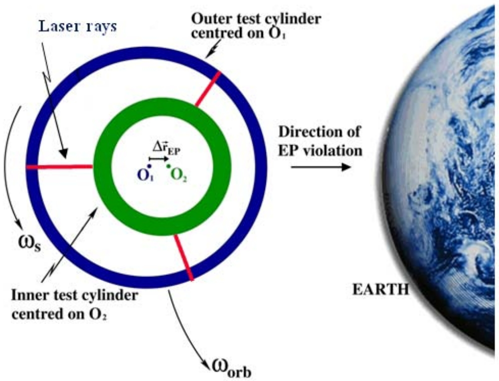
Integration time for 10^{-17} EP sensitivity and SNR=2: ~ 38 minutes!!!!

Damping noise from residual pressure is of the same order, noise force doubles, integration time grows by factor 4: ~ 2.5 hr total!!!

Macroscopic test masses weakly coupled and rapidly rotating drastically abate thermal noise!!!

[Pegna et al., in collaboration with JPL, submitted to PRL, 2011]

GG: Why laser metrology from JPL to replace main capacitance sensors?



GG has very low thermal noise and can detect an EP violation to 10^{-17} in 1 and half our. To exploit that that it needs a read out which:

- has extremely low noise
- is very sensitive to displacements in differential mode and almost insensitive to those in common mode
- disturbs the test masses (and affects their assembly) as little as possible

Laser metrology is the answer:

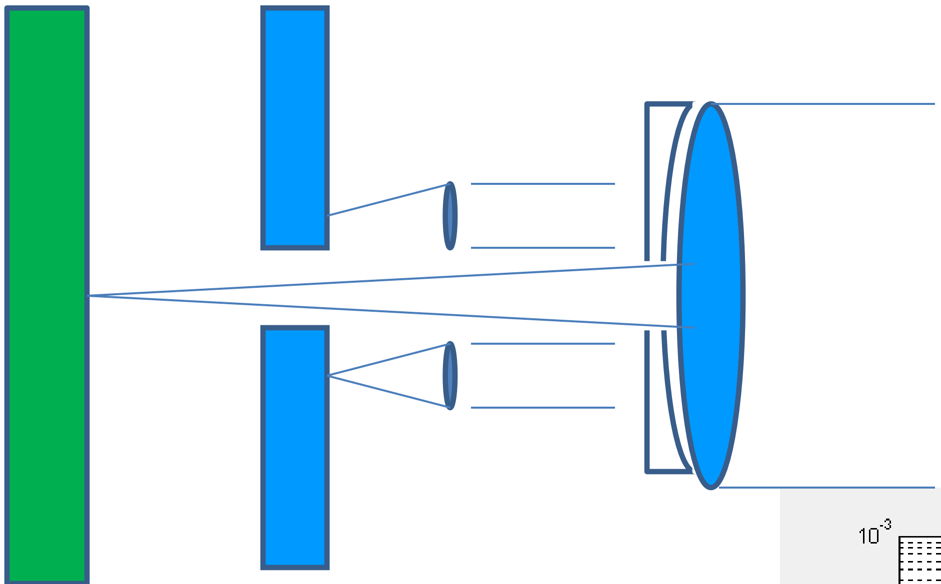
- It is far more sensitive than cap sensors
- Laser metrology is liner, hence large common mode motions do not give rise to false signals when two measurements are subtracted.
- Only light is deposited on the test masses (for the rest, they are totally "undisturbed" by the experimentalist during measurements …)

…+ allows larger gaps between the test masses (cap sensors would loose sensitivity), hence effects of electric charge patches are highly reduced and it is easier to get rid of them – larger gaps reduce other disturbances too **(GG TMs still have capacitors & springs to manage experiment initialization after unlock)**

Basic concept of GG laser gauge

Inner test mass

Outer test mass



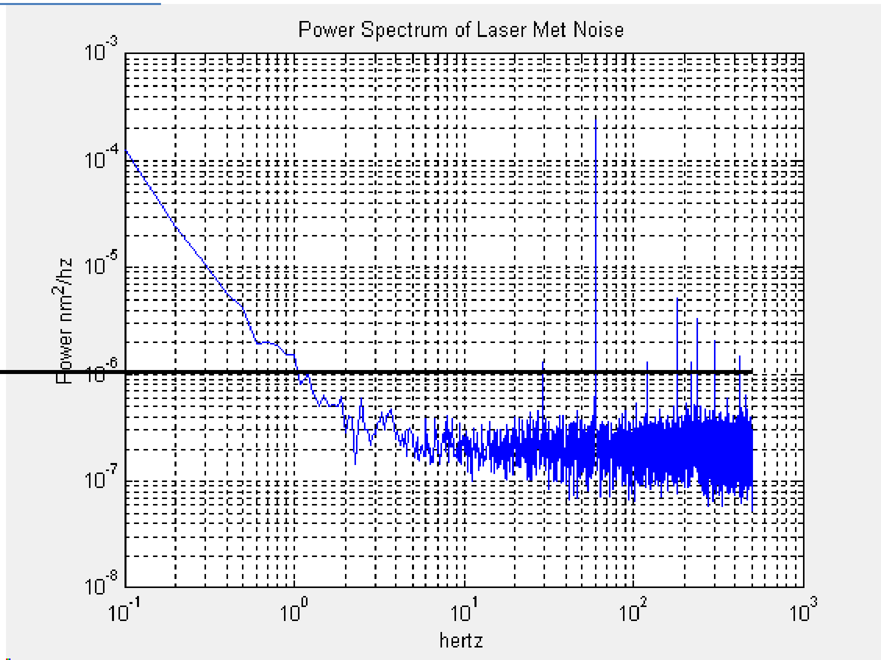
Positive lens focus
Light on inner mass

Negative lens with hole
Removes focus term
Smaller lenslets focus
light on outer testmass

Polished/coated surfaces

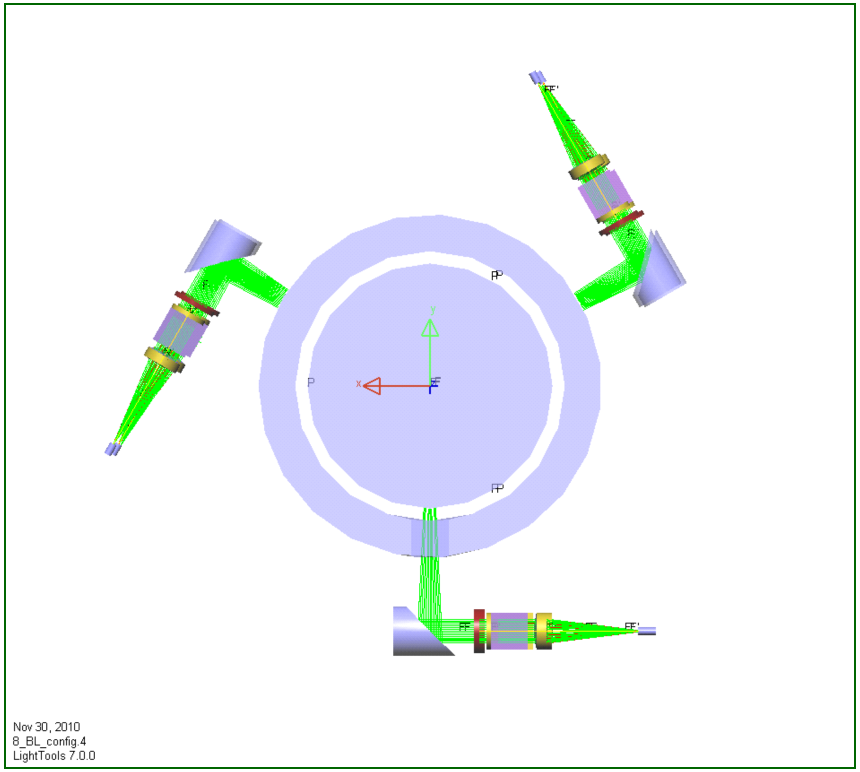
The sensitivity of the laser gauge is independent of the gap between the two test masses.

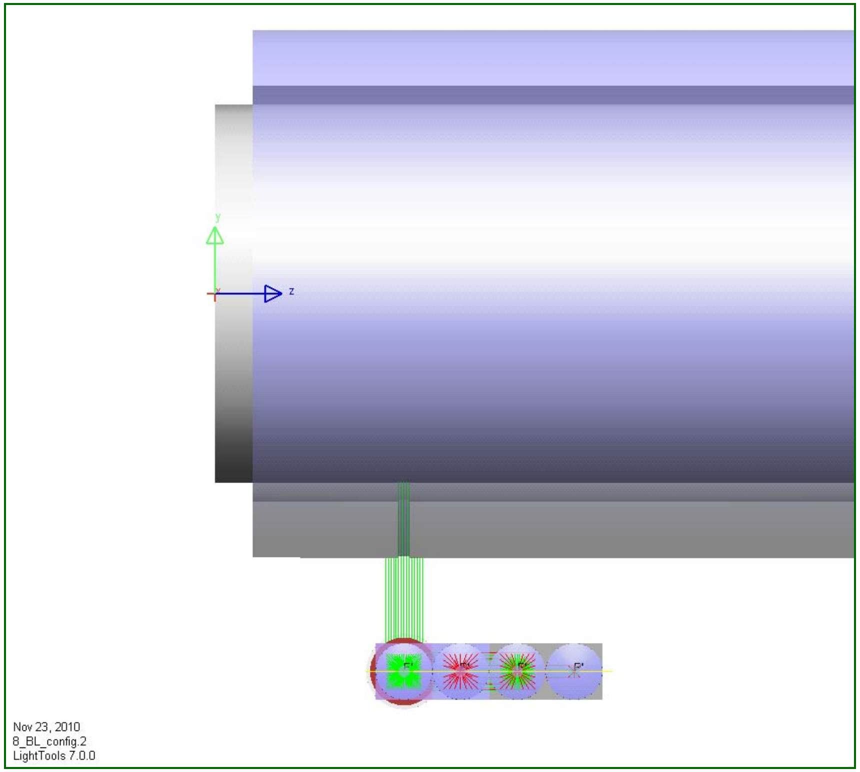
1 pico-m/rthz



GG laser gauge configuration concept (I)

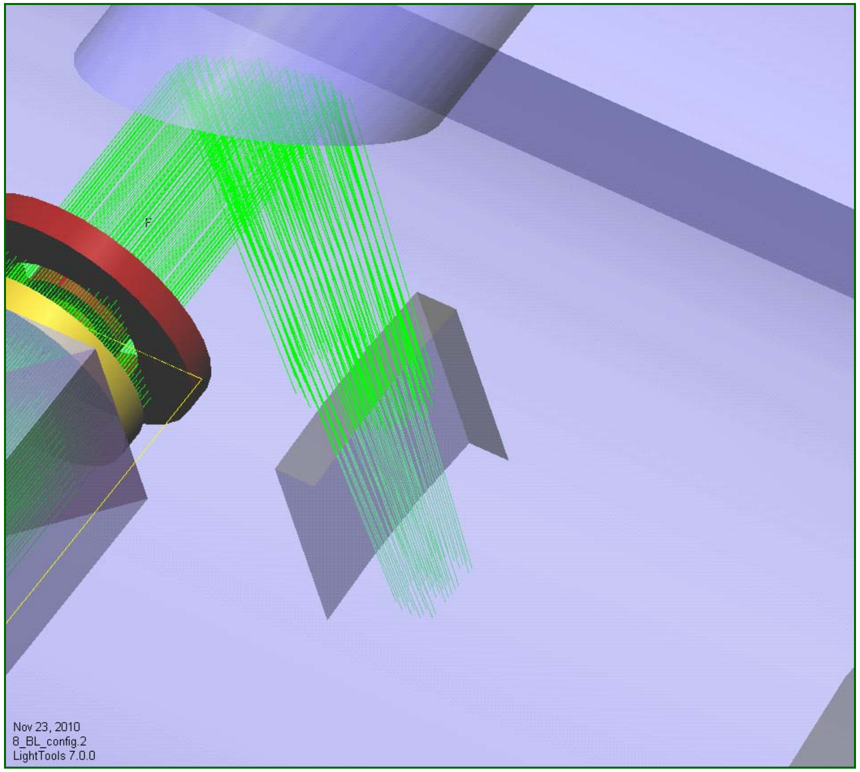
- Total six (6X) gauges, symmetrically distributed at two ends of the proof masses
- Provide 4 DOF measurement between inner and outer cylinders
- Partial redundancy

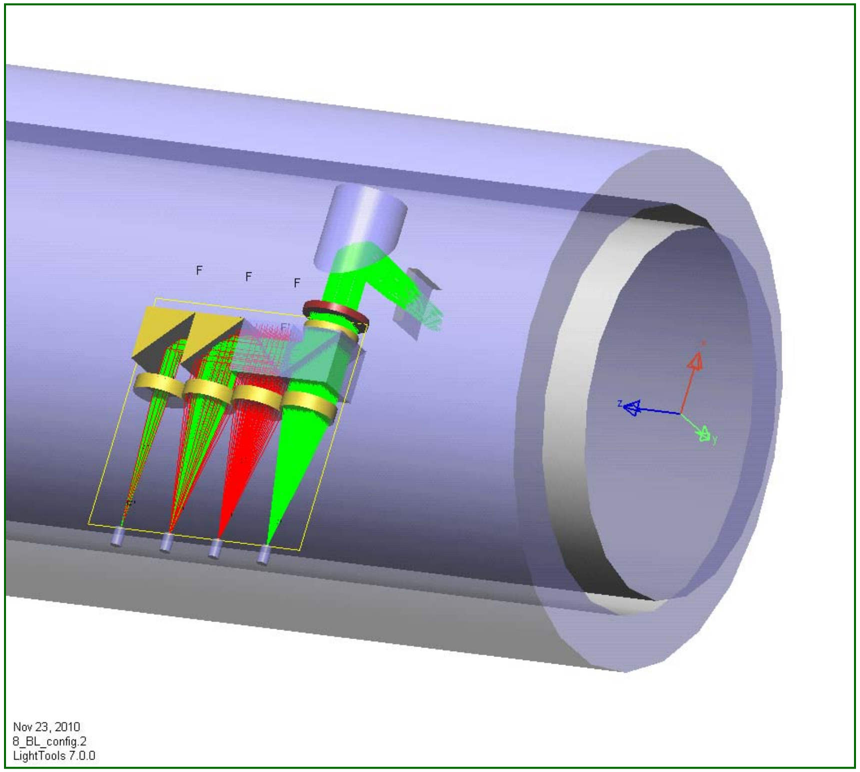




GG laser gauge configuration concept (II)

- Spatial split, between inner and outer cylinders.
- Outer cylinder has slot/holes
- Reflective patches on both inner and outer cylinders
- Heritage: SIM, PDAS, etc.





GG: Performing signal null checks with no additional accelerometer on board

If GG meets its target sensitivity and measures a non zero signal, is it possible to prove that it is EP violation (new force of Nature) and not a disturbance accountable with known Physics?

With very small thermal noise + laser metrology, the answer is yes:

- With 2.5 hr integration time for SNR=2, GG can make a full measurement to 10^{-17} in 1d (15 orbits)
- GG has extremely high spin energy and its axis (normal to the sensitive plane) is fixed in inertial space during the mission
- Instead, being in sun-synchronous orbit the normal to the orbit precesses around the Earth axis by 1^{o} per day, so the angle ϑ between the spin axis and the orbit normal changes daily. With no attitude control GG can have $-40^{o}<\vartheta<+40^{o}$ in 80 days, with an EP measurement each day….

EP violation signal and the most dangerous disturbances which need to be distinguished from it have their own specific signature as function of ϑ**,** which allows them to be very clearly distinguished……

[Nobili et al., in press]

What is around the test masses: top view

FEEP Propulsion

Nitrogen Propulsion

P/L Equipment

AOCS

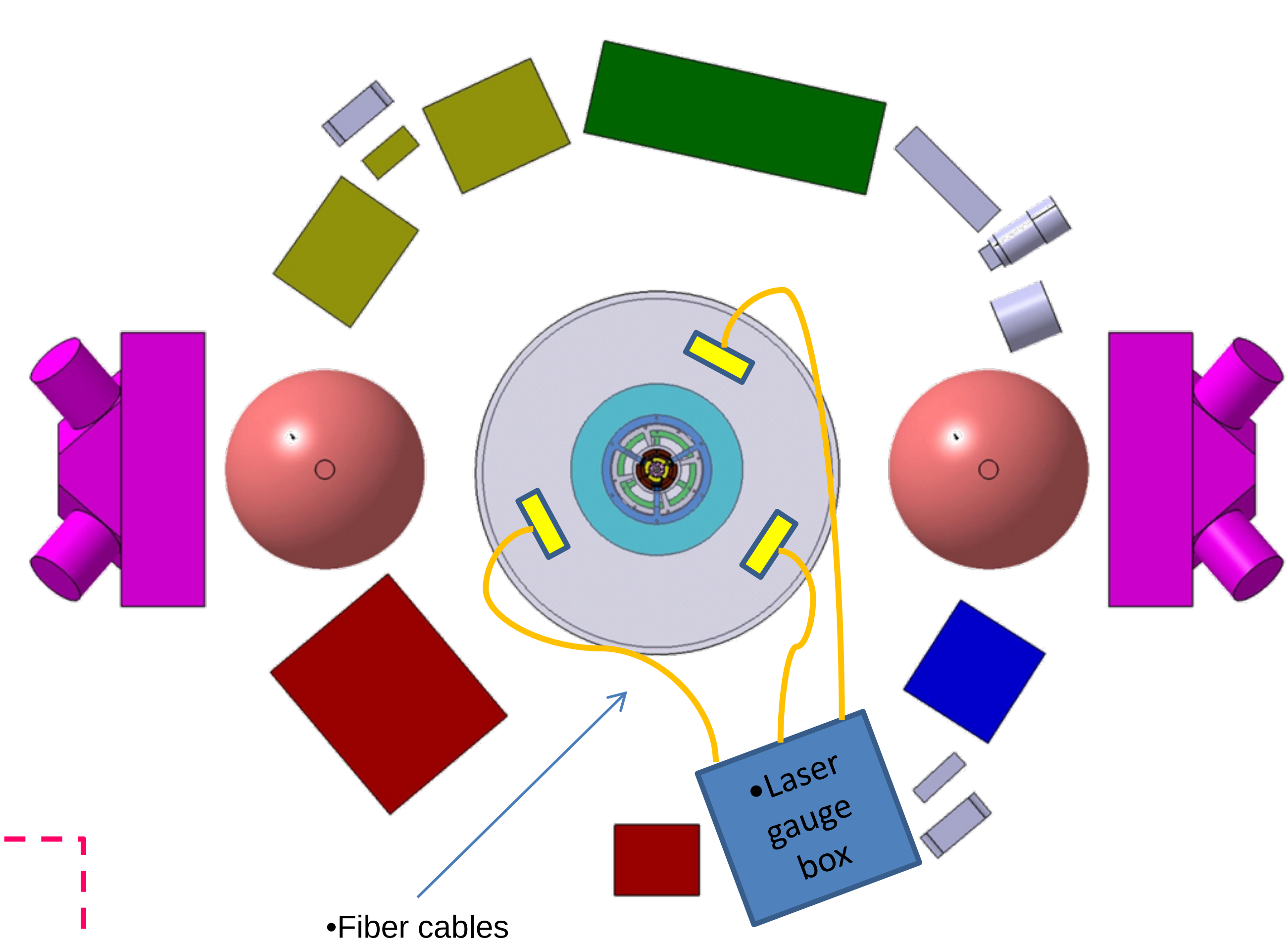
Power

Communications

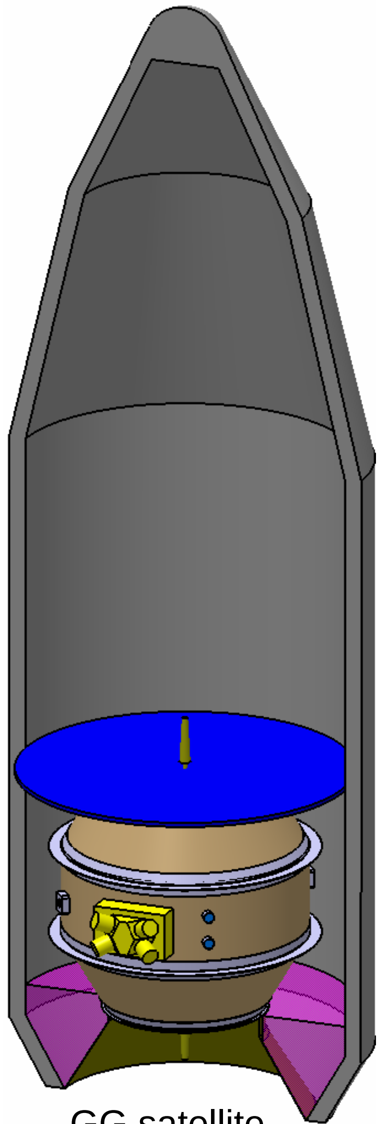
Data handling

- **Laser gauge box**
- **Beam launchers**

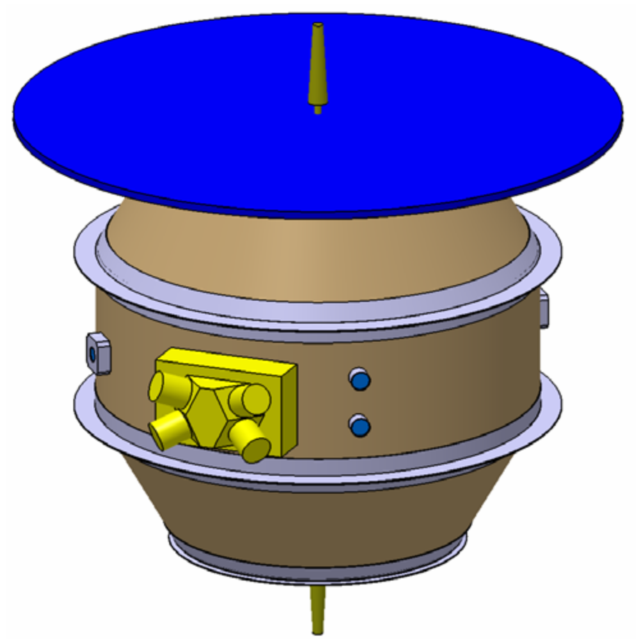
(6X)



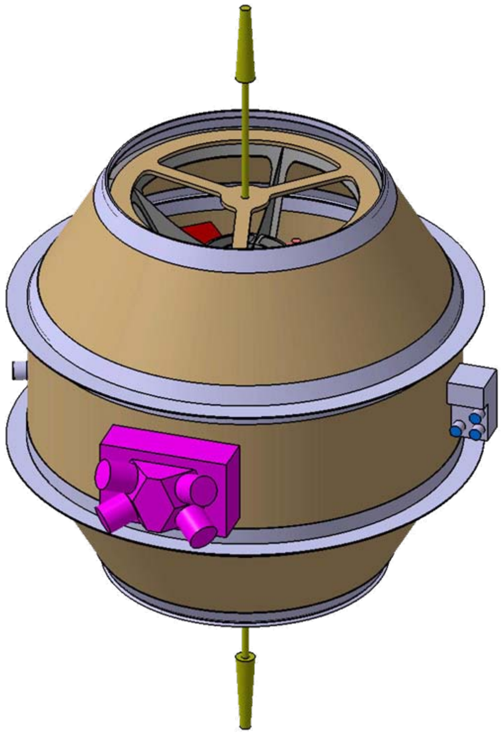
The GG satellite



GG satellite
inside Taurus
fairing



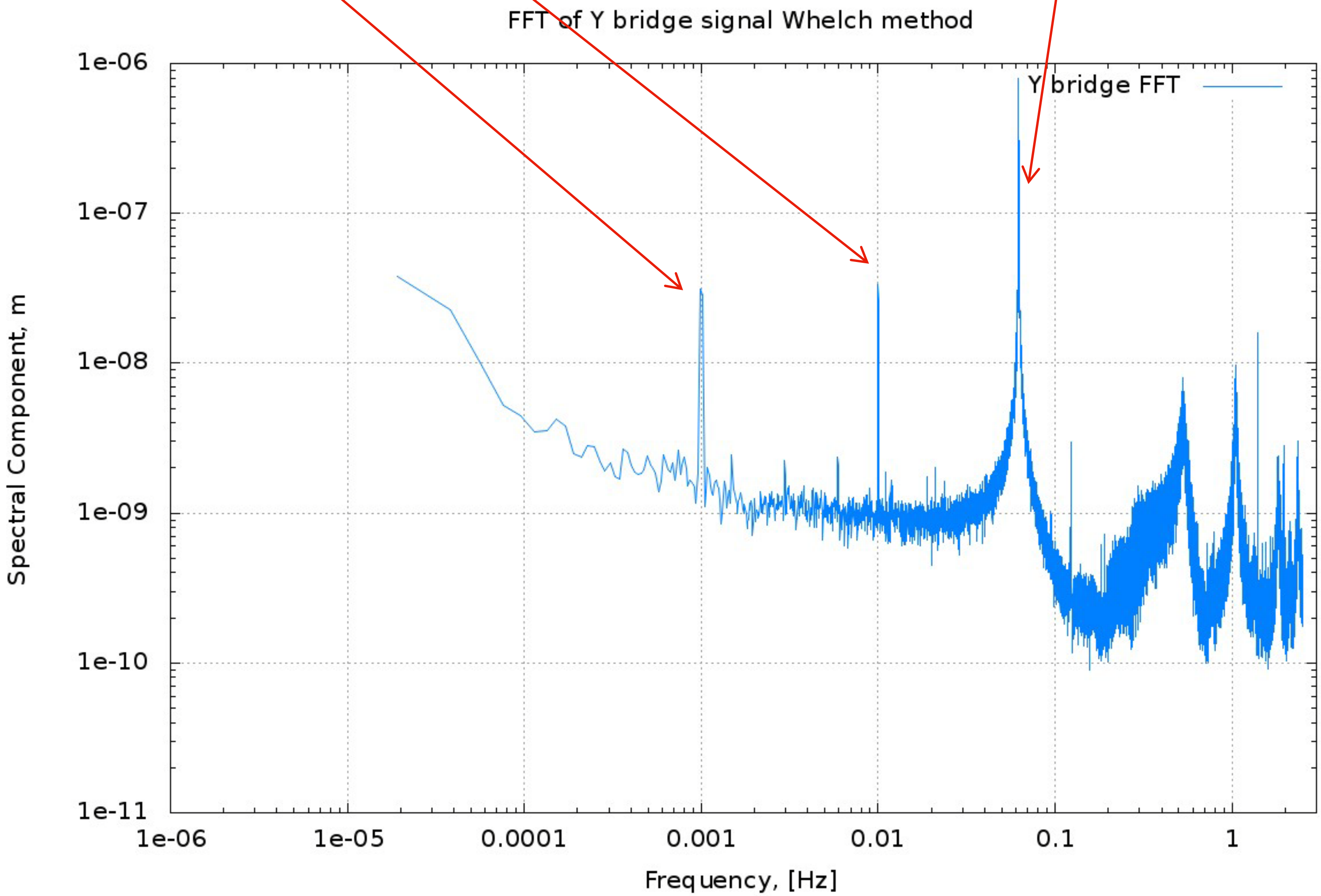
GG satellite



GG satellite without
solar panel

GGG: Evidence that signals above resonance are not attenuated (I)

GGG not rotating: 2 force signals of same amplitude applied in the lab frame at frequencies 0.001 Hz and 0.01 Hz), below the natural frequency 0.06 Hz



GGG: Evidence that signals above resonance are not attenuated (II)

GGG spinning at 0.19 Hz with natural frequency 0.1 Hz : same signals applied at 0.01 Hz and 0.001 Hz in the lab frame.

Because of the lower natural frequency they should both produce smaller effects by the ratio of the natural frequencies squared: $(0.1/0.06)^2=2.78$

Because of rotation they are upconverted above the natural frequency, and in typical 1D forced oscillators they would be attenuate another factor.

So, we should see a total attenuation by a factor 8, which we do not see in GGG

(we see only the expected factor 2.78)

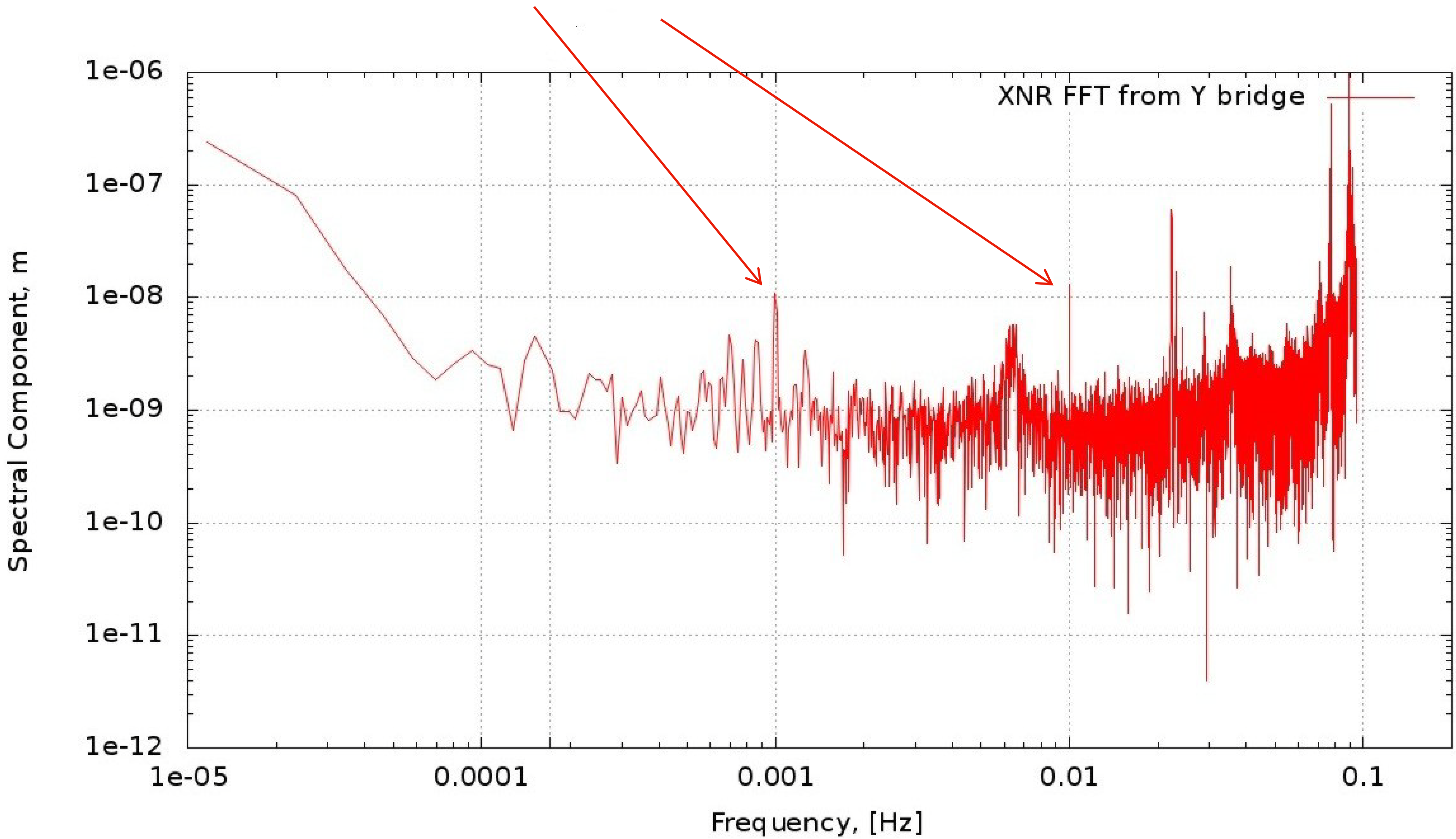
NOTE: in space the advantage is much bigger because it is possible to spin much faster than the natural frequency with no attenuation of the target EP violation signal!!

[Pegna, Nobili et al., to be submitted to PRD]

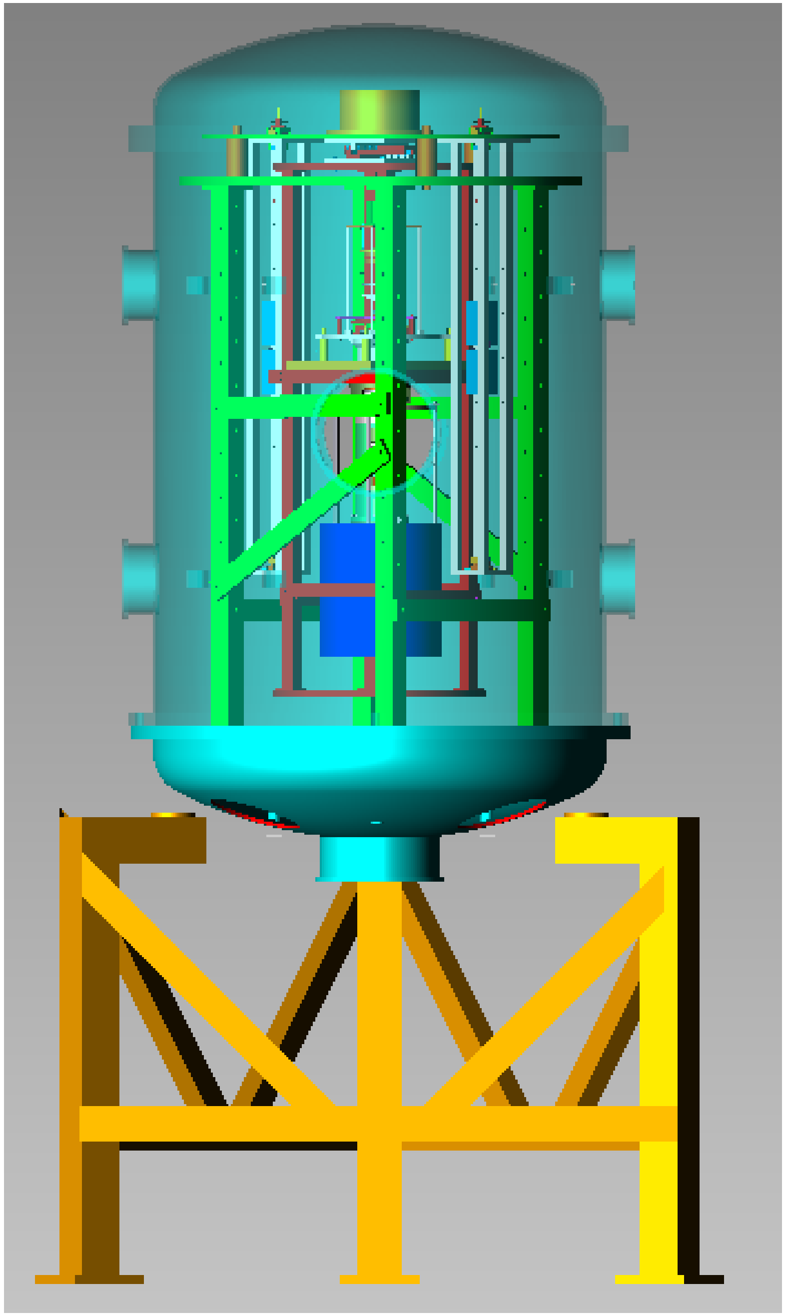
GGG: Evidence that signals above resonance are not attenuated (III)

We see only the factor almost 3 reduction expected by the slightly higher natural frequency, certainly not a factor of 8!!!

GG/GGG has the unique property that a low frequency signal can be modulated above the natural frequency (with great advantages) without being attenuated!

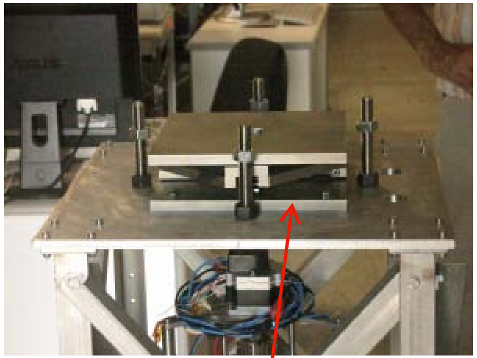


sGGG (ASI funding) (I)



sGGG (II)

sGGG (III)

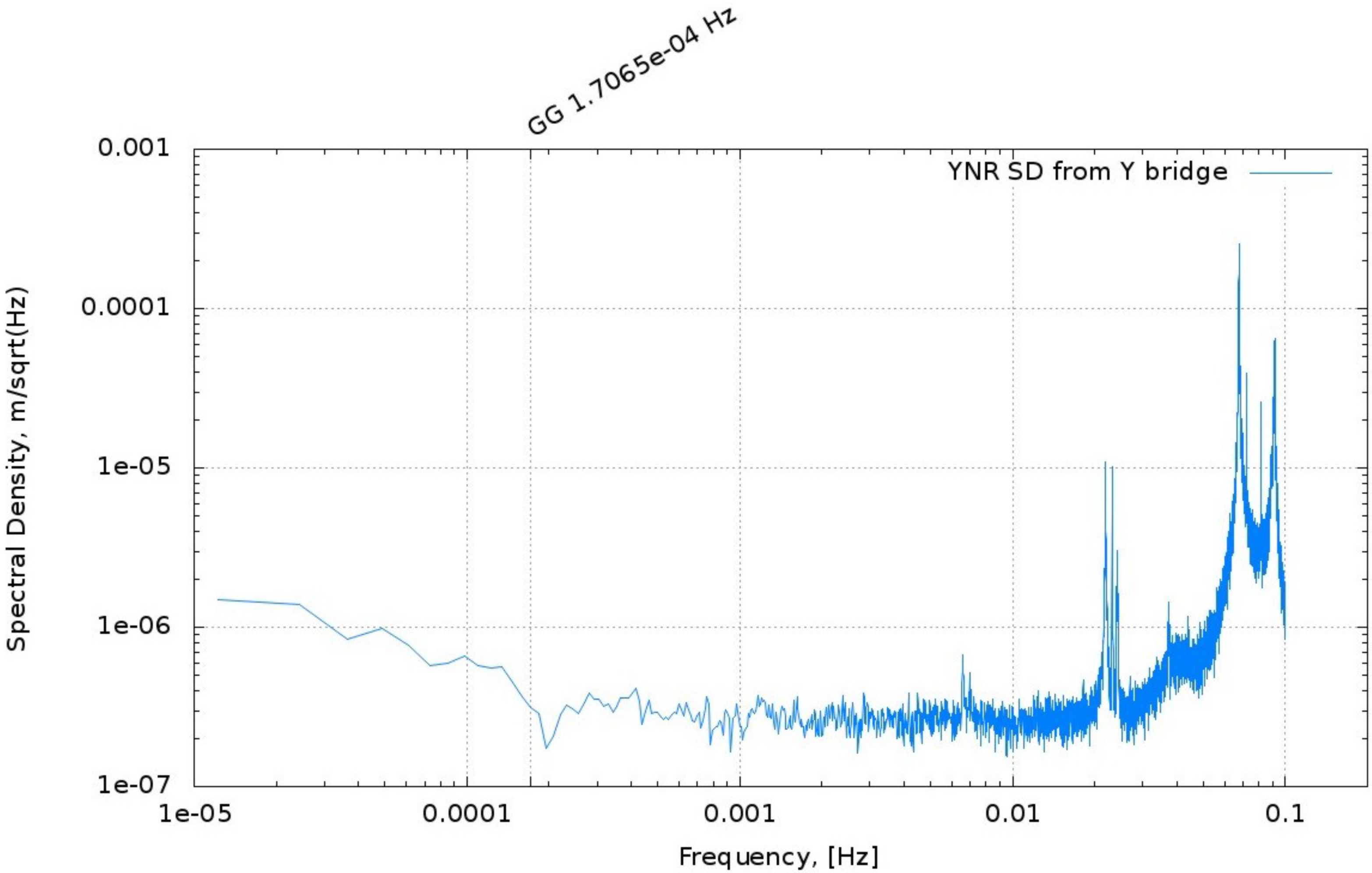


2D laminar suspension (not rotating)

First run of sGGG (end 2010)

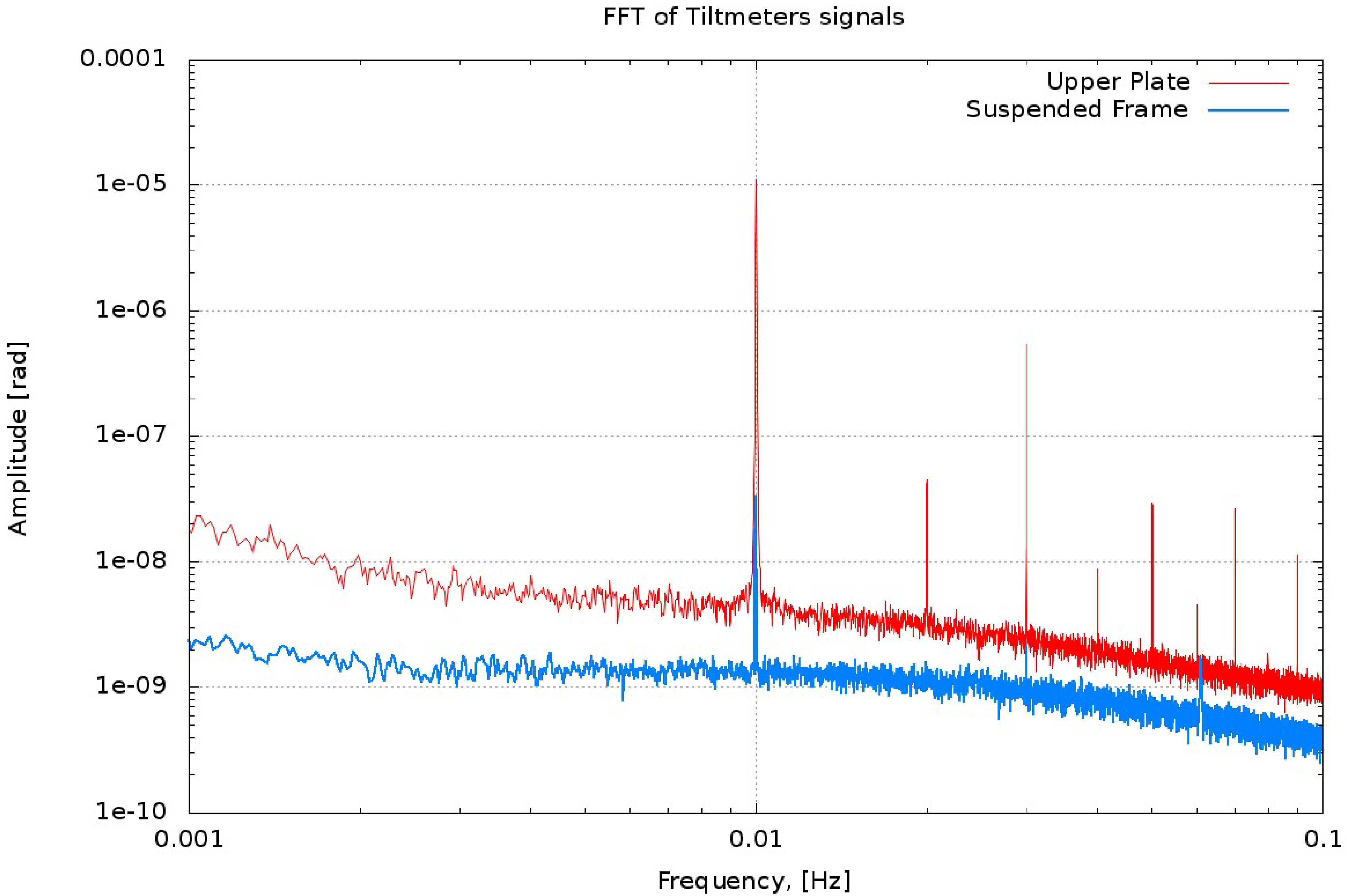
Factor 10 improvement from non-suspended system + spectrum much more flat at low frequencies (passive tilt attenuation much better than active tilt control)

@ GG orbital frequency: 3e-7 m/√Hz (sensitivity to differential acceleration of about 10^{-11}g)



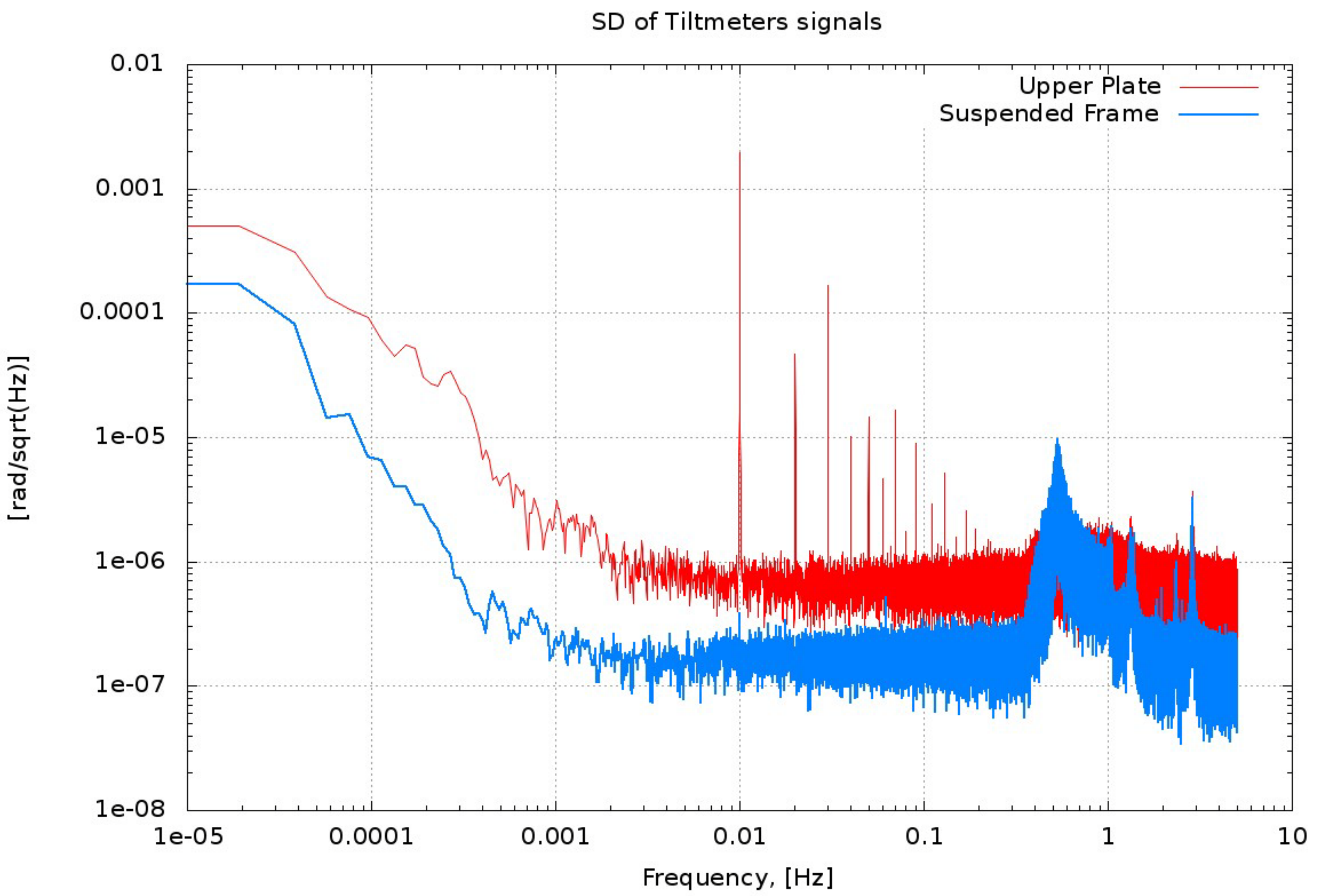
Tilt attenuation by 2D laminar suspension (I)

Attenuation factor 300 measured by applying strong tilt to the frame rigid with the chamber , zero spin (.. limited by cables connecting fixed to suspended frame)



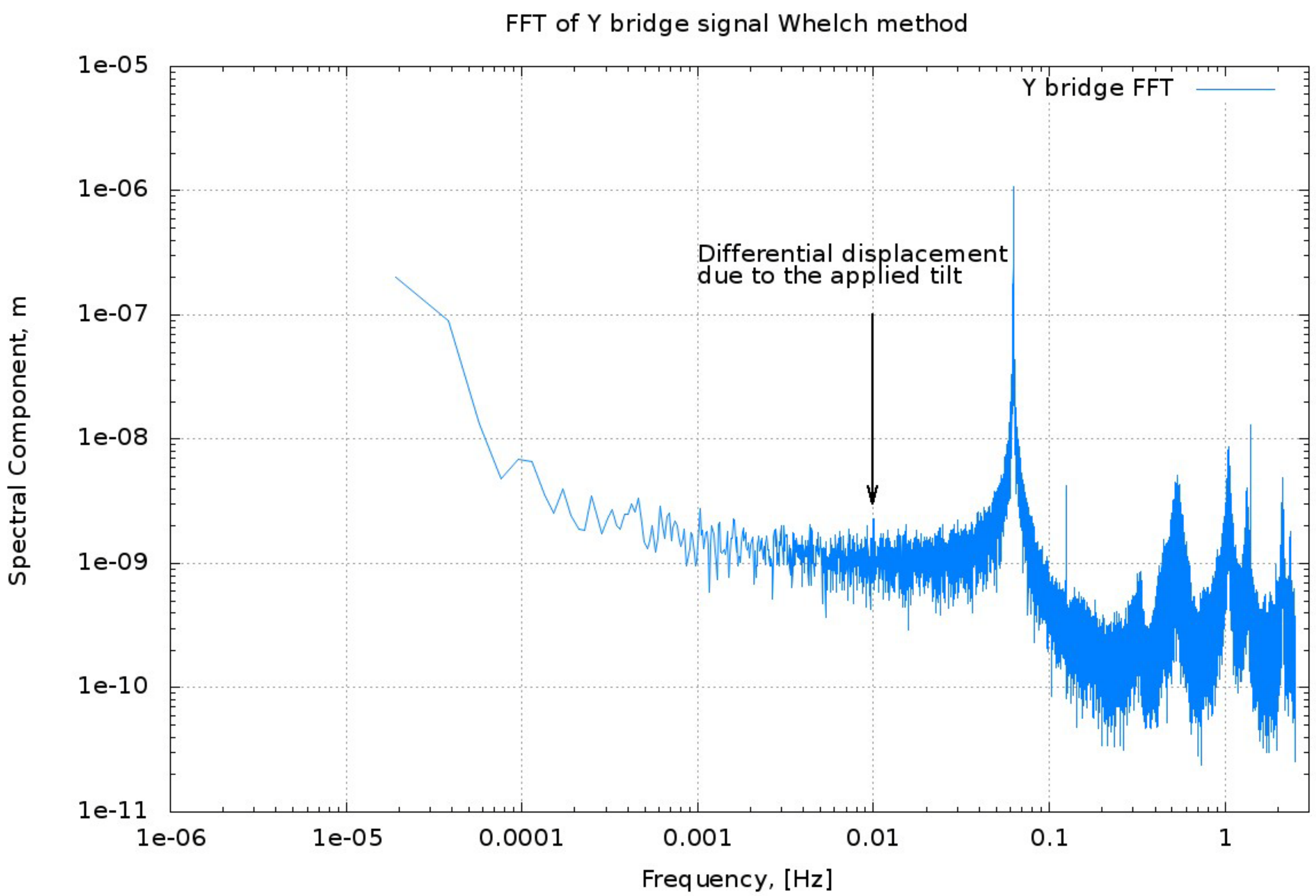
Tilt attenuation by 2D laminar suspension (II)

Attenuation factor **5000** measured after using thin cables



Tilt attenuation by 2D laminar suspension (III)

After attenuation by factor 5000 the effect of the applied tilt is visible as a relative displacement of the test masses just above noise



sGGG simulator under training

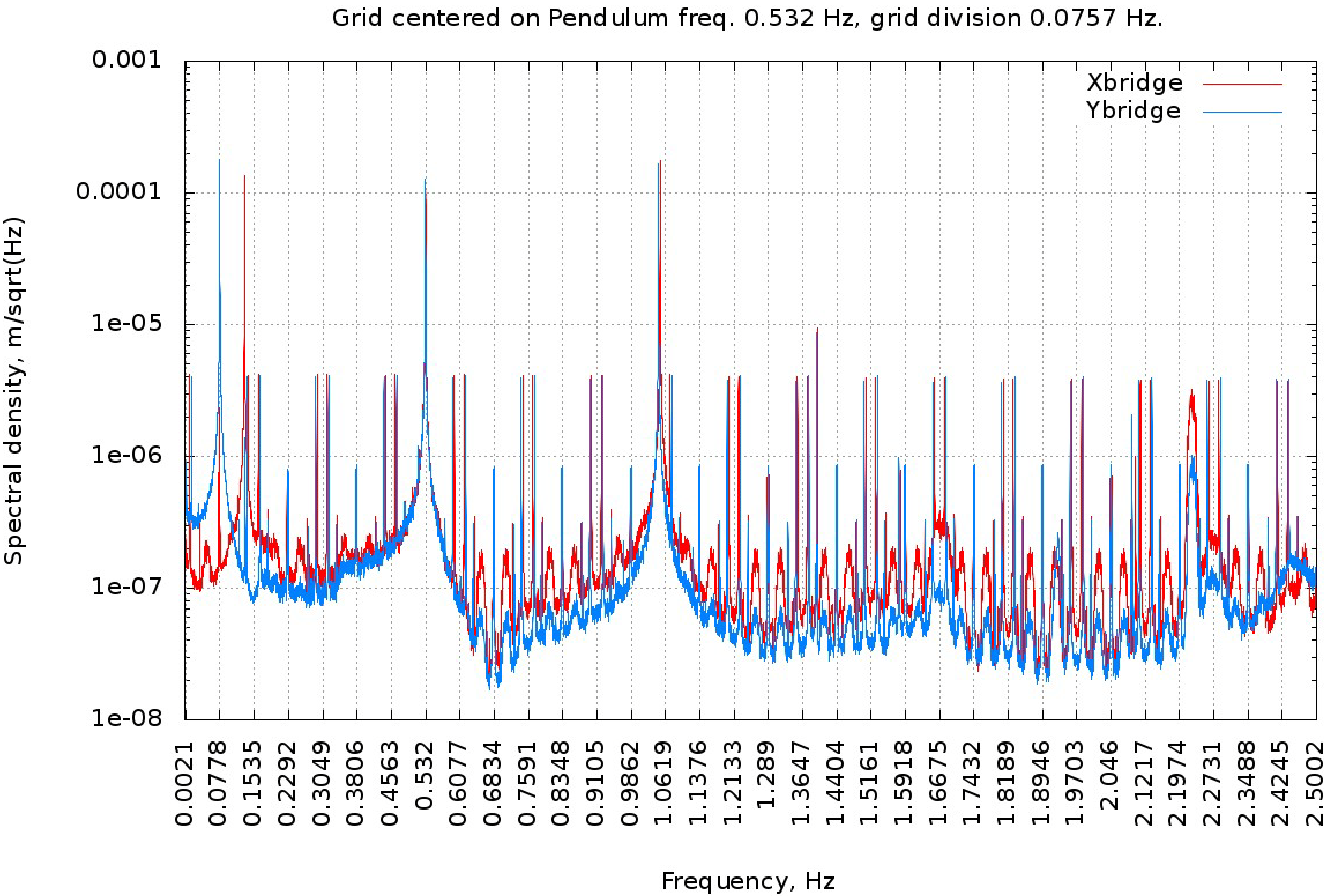
A simulator of sGGG has been set-up, based on the engineering construction drawings of the system. It is written in SimMechanics

The non rigid components of the system are implemented by forcing the simulator to match the measurements.

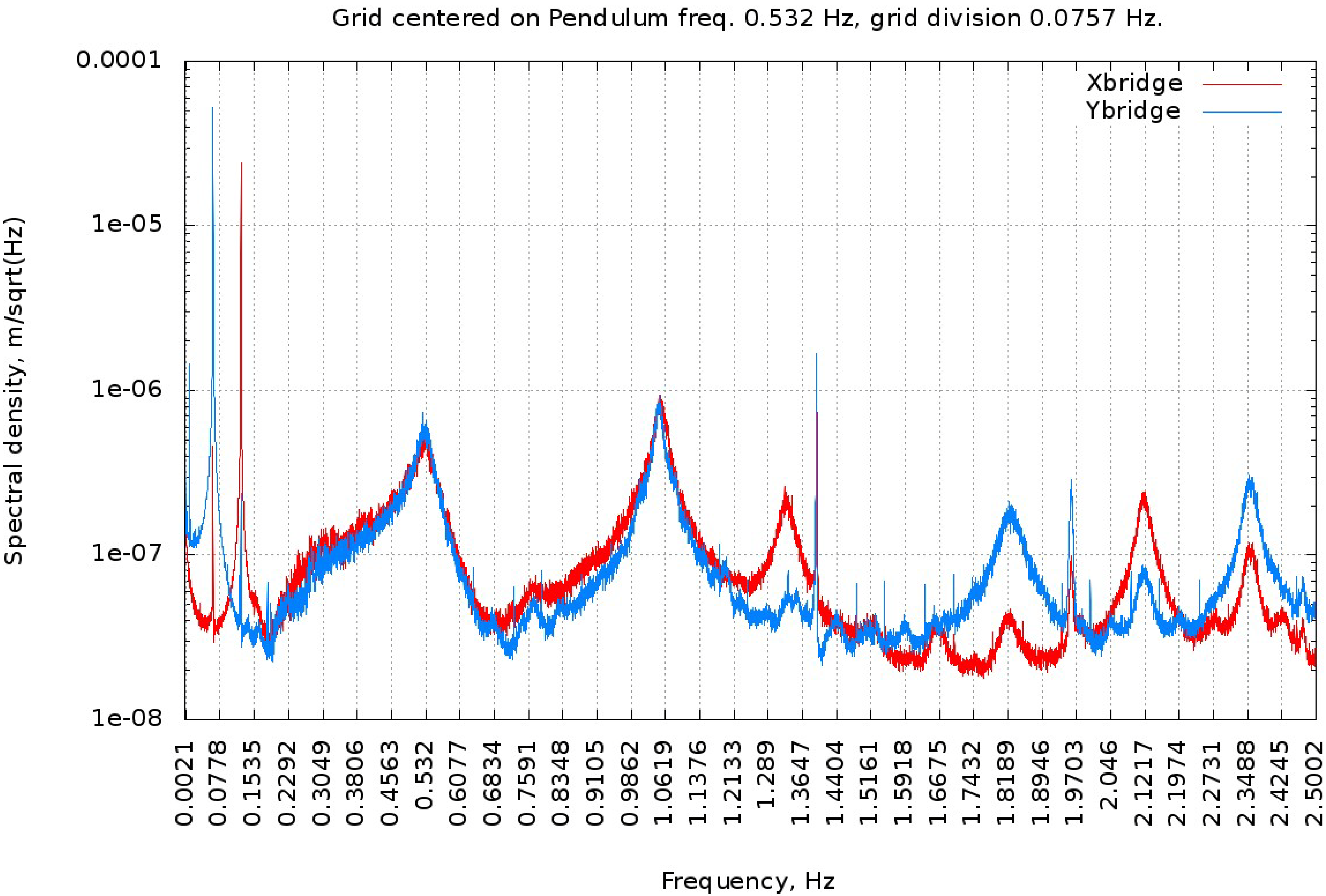
- it does reproduce all the natural frequencies
- It does reproduce the observed tilt attenuation factor

It can be used to infer effects which are hard to measure (e.g. effects of horizontal accelerations) or to check the effects of hardware changes in order to establish if they are worth implementing

Reducing energy coming from the pendulum motion of the suspended frame (I)

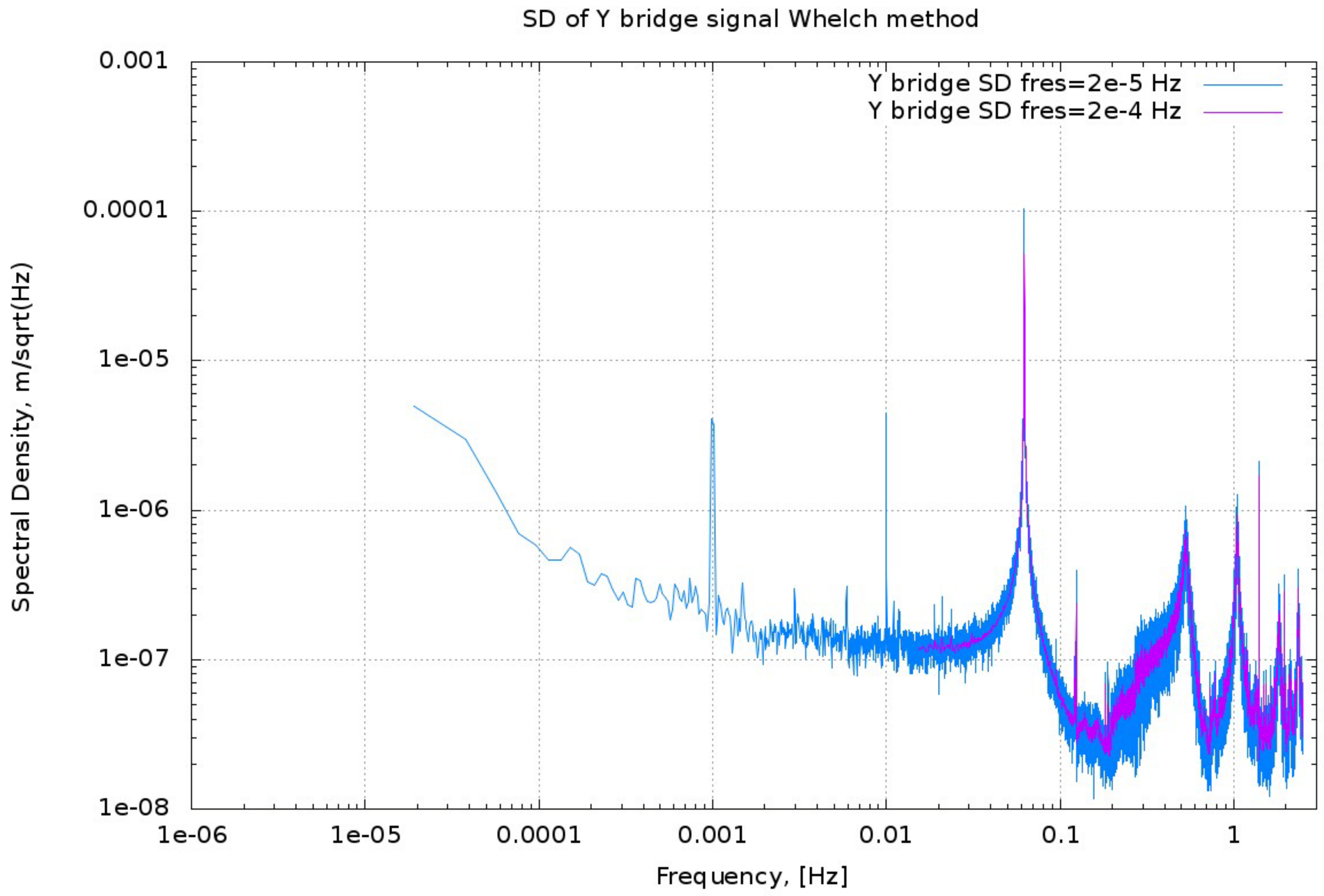


Reducing energy coming from the pendulum motion of the suspended frame (II)



Read-out electronic noise and advantage of spin

At 0.2 Hz electronic noise is about 3e-8 m/sqrt(Hz), so when spinning at 0.2 Hz it is about 1 order of magnitude smaller than current noise at the low frequency of interest (1.7e-4 Hz). So, it is not yet the limiting noise – it will be soon, so new electronics is under construction



Hunting low frequency (1.7e-4 Hz) noise sources

- Horizontal seismic accelerations: under study with simulator: most probably not yet a limitation but common mode effects may be larger
- Uniformity of rotation: demultiplied stepper motor under testing
- Rigidity of connection from rotor to laminar suspension (can have low frequency changes if not rigid enough; current frame questioned)
- Ball bearings: dust or small defects of balls may produce low frequency motion of shaft (may be the main culprit) .. Move to magnetic bearings…
- Small leakage from vacuum chamber (gives rise to low frequency disturbance): simple test proposed by Erseo Polacco will be done soon

Another factor 10 improvement not too far away

But to reach 10^{-15} g sensitivity we are heading towards a smaller rotor (test cylinder 1 kg each: weaker coupling, higher acceleration sensitivity) with optical read out (from JPL, though not the laser gauge to be used for GG in space; low noise and differential) and low noise motor and bearings .

Persone-Pisa

A. Nobili	PA	100%
T.R. Saravanan	PhD Student	100%
G. Mengali	PA	70%
F. Pegoraro	PO	40%
P. Paolicchi	PA	40%
M.L. Chiofalo	PA	40%
D.M. Lucchesi	Ricercatore INAF	40%
G. F. Gronchi	RU	40%
G. De Salvo	Fulbright Fellowship	75%
Totale ricercatori equivalenti (FTE)		5.4
Numero totale ricercatori		9

Richieste alla Sezione: 2 settimane uomo di Andrea Basti per passaggio e controllo di disegni fatti da lui o ai quali ha contribuito alla ditta esterna che paghiamo per fare i disegni costruttivi di cui abbiamo bisogno

Persone-Bologna

P. Baldi	PO	60%
P. Tortora	PA	50%
G. Casula	Ric-Ist. Naz. Geofisica	40%
(F. Palmonari	PO in pensione	-)
(S. Focardi	Professore emerito	-)
Totale ricercatori equivalenti (FTE)		1.5
Numero totale ricercatori		3

Richieste 2012 (KE)

Missioni interno	Missioni estero	Materiale consumo	Materiale inventariabile	Costruzione apparati	**Totale**
PI 2.5	6 Due settimane al JPL 1.5 Partecipazione ad 1 convegno internaz in Europa (con presentazione di talk)	3 Materiale da laboratorio		17 Modifiche hardwaare (complete di disegni costruttivi) per implementazione del read-out ottico proposto da JPL 5 Set di 3 sospensioni in CuBe lavorate in elettroerosione	35
BO 2.5	0	0	0	0	2.5
Totale					37.5

NOTA: le richieste finanziarie corrispondono al MINIMO necessario per usufruire della collaborazione con il JPL (le cifre NON sono state incrementate in vista di tagli lineari)

Abatement of thermal noise in mechanical oscillators with rapidly rotating test masses

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[2] *Dept. of Physics "E. Fermi", University of Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy*
[3] *Jet Propulsion Laboratory, California Inst. of Tech., 4800 Oak Grove Dr., Pasadena, CA 91109*
[4]*Thales Alenia Space Italia, Strada Antica di Collegno 253, 10146 Torino, Italy*
(Dated: 7 June 2011)

Mechanical oscillators are sensitive to very small forces, the sensitivity improving with the inverse of the natural oscillation frequency squared. For low frequency effects it is crucial to up-convert the signal to higher frequency. This is achieved by rotating the oscillator. We show that for a 2-dimensional oscillator rotating at a frequency much higher than the natural one, thermal noise competing with the signal is abated so that the integration time required for the signal to emerge from thermal noise is reduced by as much as the ratio of the two frequencies. Traditionally, the rotation frequency has been kept below the natural one because otherwise the signal would be attenuated as the rotation frequency squared. This has limited the advantage of rotation. We show that, provided the oscillator is allowed to move in a plane, it can rotate much faster than its natural frequency without the signal being attenuated. These findings indicate that test masses weakly coupled in 2-D and rapidly rotating can play a major role in very small force physics experiments.

Physics experiments for the measurement of small forces are ultimately limited by thermal noise. Once all systematics are reduced below the signal –and if read out noise is not a limitation– it sets the length of the integration time required for the signal to emerge above thermal noise. A factor 10 better sensitivity –i.e. a 10 times smaller force to be detected– requires an integration time 100 times longer, which makes reduction of thermal noise a must if extremely small forces are to be detected.

Consider a 2-D harmonic oscillator made of two point-like test bodies of reduced mass μ coupled by a spring of stiffness k in both directions of the plane. The general solution is an elliptic orbit with the center in the common center of mass of the bodies, which can be decomposed into the sum of two simple harmonic motions with $\omega_n = \sqrt{k/\mu}$ the frequency of *natural* (or *proper*) oscillations of the test masses relative to each other in each direction.

The oscillator is designed to be sensitive to very small forces acting between the masses in their plane of motion. Therefore, it has a very low natural frequency ω_n (because the sensitivity improves as ω_n^{-2}) and employs springs of very high mechanical quality (i.e. losses are very small). It is also operated in vacuum at low residual pressure in order to reduce damping resulting from Brownian motion.

Such a system is dominated by internal damping. According to Nyquist fluctuation-dissipation theorem, in the frequency domain the Power Spectral Density (PSD) of the thermal noise force is given by:

$$< |\hat{F}_{th}(\omega)|^2 >= 4K_BT\gamma(\omega) \tag{1}$$

with K_B the Boltzmann constant, T the thermal equilibrium temperature and $\gamma(\omega)$ the damping coefficient which, for systems dominated by internal damping has been found to be frequency dependent and given by [1]:

$$\gamma(\omega) = \frac{k\phi(\omega)}{\omega} \tag{2}$$

where ϕ is known as *loss angle* (its modulus is the inverse of the mechanical quality factor Q) which also depends on the frequency ω, albeit mildly. The name *loss angle* is because in the presence of losses the displacement always lags the applied force by the angle ϕ ($\phi(\omega)$ is an odd function). (2) is verified experimentally and the divergence at zero frequency is a known issue of no practical relevance ([1], Sec. VII).

Let ω_{signal} be the frequency of the very small force to be sensed by the oscillator, typically smaller than its natural frequency ($\omega_{signal} < \omega_n$) because in the opposite case the effect of the force would be attenuated by the oscillator as $1/\omega_{signal}^2$ (we do not consider here the particular case in which the signal is resonant with the oscillator). If all systematics and other noise sources have been made smaller than the signal, the experiment is ultimately limited by thermal noise at the frequency of the signal. Because of the frequency dependence (2), from (1) the *relevant* thermal noise random force (i.e. its component acting on the test masses at the same frequency as the signal) after an integration time t_{int} is:

$$\begin{aligned} \mathfrak{F}_{th}(\omega_{signal})|_{t_{int}} &\simeq \sqrt{\frac{4K_BT\mu\omega_n^2\phi(\omega_{signal})}{\omega_{signal}}}\frac{1}{\sqrt{t_{int}}} = \\ &= \sqrt{4K_BT\mu\omega_n\phi(\omega_{signal})}\frac{1}{\sqrt{(\omega_{signal}/\omega_n)}}\frac{1}{\sqrt{t_{int}}} \end{aligned} \tag{3}$$

in which the factor $\omega_{signal}/\omega_n < 1$ has been singled out to stress the fact that the lower is the frequency of the signal compared to the natural frequency of the system, the longer is the integration time required to bring thermal noise below the signal. Since the need for high sensitivity requires oscillators with very low natural frequency, and this must be higher than the signal frequency, high thermal noise –and consequent long integration time– appear to be serious limitations to the measurement of very small forces.

The difficulties of detecting low frequency effects can be mitigated by up-converting the signal to higher frequency. This is typically achieved by rotating the mechanical oscillator at a frequency faster than that of the signal. Let us therefore consider a 2-D harmonic oscillator, with test bodies of equal mass m for simplicity, rotating around an axis perpendicular to its a, b sensitive plane with angular velocity ω_s with respect to the inertial frame whose x, y plane coincides with the sensitive plane of the oscillator (Fig. 1). The signal is at frequency ω_{signal} in the inertial frame and it is $\omega_{signal} \ll \omega_s$.

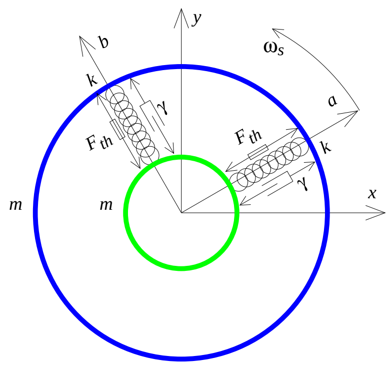

FIG. 1: Sketch of the 2-D rotating oscillator for which thermal noise is evaluated. The proof masses are concentric and rotate –together with the springs– at angular velocity ω_s. They are assumed for the moment as perfectly centered on the rotation axis. The springs are modeled as ideal springs of elastic constant k and zero length at rest; to each spring is associated a co-rotating thermal noise force generator F_{th} and an ideal noiseless damper γ. x, y is the inertial frame; a, b is the rotating one.

For the oscillator of Fig. 1 we study the effect on the relative motion of the test masses of the force due to thermal noise when the system is in thermal equilibrium at temperature T, with the purpose of assessing its relevance at the frequency of the signal.

We express the motion of the system, subject to the mechanical thermal noise force of the rotating springs, in the inertial x, y reference frame in the frequency domain and in matrix form as follows:

$$\mathbf{D}(\omega)\hat{\vec{r}} = \mathcal{F}(R(\omega_s t)\vec{F}_{th}(t))(\omega) \quad (4)$$

where $\mathbf{D}(\omega)$ is the dynamical matrix of the equations of motion of the system, $\mathcal{F}$ is the Fourier transform operator, $\vec{F}_{th}(t)$ is the thermal noise force due to losses in the rotating springs, and $R(\omega_s t)$ is the 2 by 2 rotation matrix of angle $\omega_s t$:

$$\begin{aligned} R(\omega_s t) &= \begin{pmatrix} \cos\omega_s t & -\sin\omega_s t \\ \sin\omega_s t & \cos\omega_s t \end{pmatrix} = \\ &= \frac{1}{2}e^{i\omega_s t}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + \frac{1}{2}e^{-i\omega_s t}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \end{aligned} \quad (5)$$

By defining:

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad (6)$$

we can write:

$$\begin{aligned} \mathbf{D}(\omega)\hat{\vec{r}} &= \mathcal{F}(R(\omega_s t)\vec{F}_{th}(t)) = \\ &= \mathbf{A}\hat{\vec{F}}_{th}(\omega+\omega_s) + \mathbf{A}^*\hat{\vec{F}}_{th}(\omega-\omega_s) \end{aligned} \quad (7)$$

where superscript * denotes the complex conjugate. We can see that the effect produced on the dynamical system $\mathbf{D}$ (in the inertial x, y frame) by the rotating thermal noise force $\vec{F}_{th}$ is a linear combination of $\hat{\vec{F}}_{th}(\omega+\omega_s)$ and $\hat{\vec{F}}_{th}(\omega-\omega_s)$. The most straightforward way to evaluate the components of the thermal noise force in the inertial frame is to write the time average of the Cross Spectral Density (CSD) matrix. Then —in the reasonable assumption of statistical independence of the different vectorial and frequency components of the thermal noise force— we get:

$$\begin{aligned} < \hat{\vec{F}}_{th}(\omega)\hat{\vec{F}}_{th}(\omega)^{\dagger} >= \frac{1}{2}\frac{4K_BTk\phi(\omega+\omega_s)}{(\omega+\omega_s)}\mathbf{A}+ \\ +\frac{1}{2}\frac{4K_BTk\phi(\omega-\omega_s)}{(\omega-\omega_s)}\mathbf{A}^* \end{aligned} \quad (8)$$

where $\hat{\vec{F}}_{th}(\omega)^{\dagger}$ denotes the transpose conjugate of $\hat{\vec{F}}_{th}(\omega)$.

Let us now consider the signal force of interest $\vec{F}_{signal}(t)$ acting on the test masses relative to each other at a very low frequency $\omega_{signal} \ll \omega_s$ in the inertial frame:

$$\vec{F}_{signal}(t) = F_{signal}(cos(\omega_{signal}t), sin(\omega_{signal}t)) \quad (9)$$

which in the frequency domain reads:

$$\begin{aligned} \hat{\vec{F}}_{signal}(\omega) &= \frac{1}{2}F_{signal} \\ \cdot &\begin{pmatrix} \delta(\omega-\omega_{signal}) + \delta(\omega+\omega_{signal}) \\ -i\delta(\omega-\omega_{signal}) + i\delta(\omega+\omega_{signal}) \end{pmatrix} \end{aligned} \quad (10)$$

The CSD matrix of the signal is then:

$$\begin{aligned} < \hat{\vec{F}}_{signal}(\omega)\hat{\vec{F}}_{signal}(\omega)^{\dagger} >= \frac{1}{2}F^2_{signal} \\ \cdot\left[\delta(\omega-\omega_{signal})\mathbf{A} + \delta(\omega+\omega_{signal})\mathbf{A}^*\right] \end{aligned} \quad (11)$$

By comparing (11) with (8) we can see that only the components of noise at the frequency of the signal, i.e. those with $\omega = \omega_{signal}$ and $\omega = -\omega_{signal}$ do compete with it. By evaluating the diagonal matrix elements of (8) at the signal frequencies we obtain the PSD of the x, y components of the noise competing with the corresponding components of the signal (11). That is, we must compare:

$$\frac{1}{2}\left[\frac{4K_BTk\phi(\pm\omega_{signal}+\omega_s)}{(\pm\omega_{signal}+\omega_s)} + \frac{4K_BTk\phi(\pm\omega_{signal}-\omega_s)}{(\pm\omega_{signal}-\omega_s)}\right] \quad \text{with} \quad \frac{1}{2}F^2_{signal} \quad (12)$$

Since we are in the condition $\omega_{signal} \ll \omega_s$, it is apparent that in (12) the dependence on ω_{signal} disappears and only that on ω_s remains; moreover, the off diagonal elements of the CSD (8) are very small. In these conditions the x, y components of the thermal noise force are almost uncorrelated and by averaging the x with the y component of the signal we gain a factor $\sqrt{2}$ in the signal-to-noise ratio. Thus, the thermal noise force competing with the signal is:

$$< |\hat{F}_{th}| > \simeq \sqrt{\frac{4K_B T k \phi(\omega_s)}{\omega_s}} \tag{13}$$

from which we can write the relevant thermal noise force after an integration t_{int}:

$$\mathfrak{F'}_{th}(\omega_{signal} \ll \omega_s)|_{t_{int}} \simeq \sqrt{\frac{4K_B T \mu \omega_n^2 \phi(\omega_s)}{\omega_s}} \frac{1}{\sqrt{t_{int}}} = \\ = \sqrt{4K_B T \mu \omega_n \phi(\omega_s)} \frac{1}{\sqrt{(\omega_s/\omega_n)}} \frac{1}{\sqrt{t_{int}}} \tag{14}$$

to be compared with (3) in order to appreciate the difference with respect to the non rotating case.

We see that the frequency of the signal is now replaced in (14) by the rotation frequency of the oscillator ω_s. One advantage is that losses at higher frequency are found to be smaller than losses at lower frequency, hence it will be $\phi(\omega_s) < \phi(\omega_{signal})$. The other major advantage is that the ratio ω_s/ω_n is considerably higher than the corresponding ω_{signal}/ω_n ratio in (3), since the whole purpose of rotation is to up-convert the signal to higher frequency. For a given force signal, the integration time needed to reduce the thermal force noise below it is inversely proportional to this ratio, so the higher the ratio, the shorter the integration time —i.e. the less the experiment is affected by thermal noise. There is nothing mysterious about this result: the energy of thermal noise is the same as at zero spin –simply, its component at the frequency of the signal is much smaller than at zero spin.

Traditional attempts at reducing thermal noise have involved cooling down the apparatus in order to reduce the thermal equilibrium temperature T; even if successful, cryogenics can reduce the integration time by a factor 10 at most, while rotation can do much better than that, especially if coupling is very weak.

An example are the rotating torsion balances used to test the Equivalence Principle by detecting the twist angle produced by tiny differential forces acting in the horizontal plane. They have been able to reach the level of thermal noise ([2], Fig. 20) and in this remarkable achievement they have found that thermal noise obeys the law ([2], eq. (57)) as predicted by [1], where the frequency involved is the modulation frequency of the signal, that is the frequency at which the balance rotates. Eq. (57) in [2] is the same as our (13). By rotating the balance with a period of about 20 minutes they have improved by a factor 70 as compared to relying on the 24-hr rotation of the Earth, reducing the integration time by the same factor.

However, torsion balances reported in [2] rotate at a frequency smaller than their natural torsion frequency (at about 2/3 of it), thus the ratio ω_s/ω_n in (14) is still smaller than unity. This is because, with the force signal to be detected acting at very low frequency in the inertial frame (either from the Earth, the Sun or the center of our galaxy), the balance is forced to oscillate at $\omega_s \pm \omega_{signal} \simeq \omega_s$: in each turn, when the balance arm is aligned with the force the signal is zero, when it is perpendicular to it the signal is either maximum or minimum. The balance is an oscillator with loss angle $\phi(\omega_s)$, natural frequency ω_n, forced at the rotation frequency ω_s, hence –if the forcing frequency is much higher than the natural one– the effect is attenuated as the inverse of the forcing frequency squared. This is why torsion balances are rotated at a frequency close to –but smaller than– their torsion frequency, which means $\omega_s/\omega_n \lesssim 1$. As a matter of fact, in ([2], Sec. 6.2.1) while analyzing ways to reduce thermal noise, it is stated that thermal noise is minimized by modulating the signal at high frequency, but a further increase of the rotation/modulation rate is not considered as an option.

Below we show how it is possible to realize a 2-D oscillator rotating at a frequency much higher than its natural one (i.e. with $\omega_s/\omega_n \gg 1$ and consequent high frequency modulation and abatement of thermal noise) without the signal being attenuated.

So far we have referred to a rotating oscillator in which the proof masses are perfectly centered on the rotation axis. In reality perfect centering is impossible; we represent such manufacturing imperfections by an offset vector $\vec{\epsilon}$ of the reduced mass μ from the rotation axis ($\vec{\epsilon}$ is fixed in the rotating frame). The equilibrium position vector is:

$$\vec{r}_{eq} = \frac{1}{1 - (\omega_s/\omega_n)^2} \vec{\epsilon} \tag{15}$$

which in the case that the rotation frequency is much larger than the natural one becomes:

$$\vec{r}_{eq} \simeq -\vec{\epsilon} \left(\frac{\omega_n}{\omega_s}\right)^2 \tag{16}$$

showing that the center of mass of the rotating body reaches equilibrium much closer to the rotation axis than it was by construction, by the factor $(\omega_n/\omega_s)^2 \ll 1$. This auto-centering phenomenon is what makes fast rotation more advantageous than the slow one. The minus sign means that for the equilibrium position to be reached the center of mass of the body must be allowed to move in the rotating plane so as to set itself antiparallel to $\vec{\epsilon}$ as required for equilibrium by (16): if constrained in 1 direction only, it would not auto-center [3].

Let us now write and solve the equations of motion around the equilibrium position in the presence of a force, like the signal, of very low frequency. In the inertial frame they read:

$$\mu\ddot{\vec{r}} + \gamma_{\omega_s}(\dot{\vec{r}} - \vec{\omega}_s \times \vec{r}) + k\vec{r} = \vec{F} \tag{17}$$

where γ_{ω_s} is the damping coefficient of the oscillator rotating at ω_s, which is very small because the oscillator has very small losses and of the type (2) because it is dominated by internal damping; $\vec{F}$ is the signal force whose frequency is so small compared to both ω_s and ω_n that we consider it as constant. In the 2-body oscillator of Fig. 1, if the bodies have equal mass m the reduced mass is $m/2$, the natural frequency is $\omega_n = \sqrt{k/(m/2)}$ with the external force acting between them. In the assumptions made ($\omega_s \gg \omega_n$ and very small internal losses), the solution of the homogeneous part of (17) is (with amplitudes and phases determined by initial conditions):

$$\begin{aligned}\vec{r}_w(t) \simeq A_0 e^{\phi_{\omega_s}\omega_n t/2}\begin{pmatrix}\cos(\omega_n t + \varphi_A)\\ \sin(\omega_n t + \varphi_A)\end{pmatrix} + \\ + B_0 e^{-\phi_{\omega_s}\omega_n t/2}\begin{pmatrix}\cos(-\omega_n t + \varphi_B)\\ \sin(-\omega_n t + \varphi_B)\end{pmatrix}\end{aligned} \tag{18}$$

showing that in the inertial reference frame the oscillator performs a combination of a forward and a backward orbital motion –known as *whirl motion*– at the (slow) natural frequency ω_n, and the radii of such orbits are exponentially decaying in the case of the backward whirl and exponentially growing in the case of the forward one. We have written the exponential behavior in terms of the loss angle:

$$\phi_{\omega_s} \simeq \frac{\gamma_{\omega_s}\omega_s}{\mu\omega_n^2} = \frac{\gamma_{\omega_s}\omega_s}{k} \tag{19}$$

which is very small because the system has very small losses; hence the forward whirl is a very weak instability. Every natural/whirl period the radius of the forward whirl grows by the fraction $\pi\phi_{\omega_s}$, hence the tangential force which produces the growth is –in modulus– $\phi_{\omega_s} kr$, which is a very small fraction of the elastic force, requiring a correspondingly small force to stabilize it. Its frequency is the natural one and does not interfere with the force signal (see [4], [5]).

In the presence of an external constant force $\vec{F}$, the equations of motion (17) show that (in the inertial frame) the body is displaced to the position:

$$\begin{aligned}\vec{r}_F(t) = \frac{1}{1 + \frac{\gamma_{\omega_s}^2\omega_s}{k^2}} \\ \cdot\left(\frac{\vec{F}_e}{k} - \frac{\gamma_{\omega_s}}{k^2}\vec{\omega}_s \times \vec{F}\right) \simeq \frac{\vec{F}}{k} - \phi_{\omega_s}\frac{\vec{\omega}_s}{\omega_s} \times \frac{\vec{F}}{k}\end{aligned} \tag{20}$$

That is, the applied force $\vec{F}$ gives rise to a displacement $\vec{F}/k$ (i.e. inversely proportional to the natural frequency squared) with a negligible additional effect (because of the very small loss angle ϕ_{ω_s}) in the orthogonal direction. In the rotating frame of the oscillator the constant displacement observed in the inertial one appears at the rotation frequency $\omega_s \gg \omega_n$, yet it is apparent that no attenuation occurs.

The general solution in the inertial frame –including the auto-centered position (16) fixed on the rotating oscillator– is:

$$\begin{aligned}\vec{r}(t) \simeq -\vec{\epsilon}(\omega_s t)\left(\frac{\omega_n}{\omega_s}\right)^2 + \frac{\vec{F}}{k} - \phi_{\omega_s}\frac{\vec{\omega}_s}{\omega_s} \times \frac{\vec{F}}{k} + \\ + A_0 e^{\phi_{\omega_s}\omega_n t/2}\begin{pmatrix}\cos(\omega_n t + \varphi_A)\\ \sin(\omega_n t + \varphi_A)\end{pmatrix} + \\ + B_0 e^{-\phi_{\omega_s}\omega_n t/2}\begin{pmatrix}\cos(-\omega_n t + \varphi_B)\\ \sin(-\omega_n t + \varphi_B)\end{pmatrix}\end{aligned} \tag{21}$$

Assume zero losses and no external force: only the first term is not zero and the solution is the auto-centered position rotating at frequency ω_s; if the force signal $\vec{F}$ is added –still with zero losses– the term $\vec{F}/k$ is not zero and the oscillator is displaced by this vector with auto-centering holding as before; finally, if small losses occur –after the backward whirl has died out, and neglecting the small effect $\propto \phi_{\omega_s}$– the forward whirl slowly grows around the displaced position at frequency ω_n. By controlling this weak instability, rotation (and signal modulation) at a frequency much higher than the natural one are achieved with no signal attenuation, and thermal noise is drastically reduced according to (14)

By overcoming a long standing limitation, rapidly rotating weakly coupled 2-D mechanical oscillators can play a major role in physics experiments for the measurement of extremely small forces.

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